

A FINITE PRESENTATION OF THE LEVEL 2 PRINCIPAL CONGRUENCE SUBGROUP OF $GL(n; \mathbb{Z})$

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ABSTRACT. It is known that the level 2 principal congruence subgroup of $GL(n; \mathbb{Z})$ has a finite generating set (see [7]). In this paper, we give a finite presentation of the level 2 principal congruence subgroup of $GL(n; \mathbb{Z})$.

1. INTRODUCTION

For $n \geq 1$, let $\Gamma_2(n) = \ker(GL(n; \mathbb{Z}) \rightarrow GL(n; \mathbb{Z}_2))$. We call $\Gamma_2(n)$ the *level 2 principal congruence subgroup* of $GL(n; \mathbb{Z})$. Note that for $A \in \Gamma_2(n)$ the diagonal entries of A are odd and the others are even.

For $1 \leq i, j \leq n$ with $i \neq j$, let E_{ij} denote the matrix whose (i, j) entry is 2, diagonal entries are 1 and others are 0, and let F_i denote the matrix whose (i, i) entry is -1 , other diagonal entries are 1 and others are 0. It is known that $\Gamma_2(n)$ is generated by E_{ij} and F_i for $1 \leq i, j \leq n$ with $i \neq j$ (see [7]).

In this paper, we give a finite presentation of $\Gamma_2(n)$.

Theorem 1.1. *For $n \geq 1$, $\Gamma_2(n)$ has a finite presentation with generators E_{ij} and F_i , for $1 \leq i, j \leq n$ with $i \neq j$, and with the following relators*

- (1) F_i^2 ,
- (2) $(E_{ij}F_i)^2, (E_{ij}F_j)^2, (F_iF_j)^2$ (when $n \geq 2$),
- (3) (a) $[E_{ij}, E_{ik}], [E_{ij}, E_{kj}], [E_{ij}, F_k], [E_{ij}, E_{ki}]E_{kj}^2$ (when $n \geq 3$),
 (b) $[E_{ji}F_jE_{ij}F_iE_{ki}^{-1}E_{kj}, E_{ki}F_kE_{ik}F_iE_{ji}^{-1}E_{jk}]$ for $i < j < k$ (when $n \geq 3$),
- (4) $[E_{ij}, E_{kl}]$ (when $n \geq 4$),

where $[X, Y] = X^{-1}Y^{-1}XY$ and $1 \leq i, j, k, l \leq n$ are mutually different.

We note that a finite presentation of $\Gamma_2(n)$ has been obtained also by Fullarton [3] and Margalit-Putman.

It is clear that the above theorem is valid in the case $n = 1$. A proof of the theorem is by induction on n . In Section 3, we will prove the case $n = 2$ of Theorem 1.1, using the Reidemeister-Schreier method. In Section 4, we will prove the case $n = 3$ of Theorem 1.1, considering a simply connected simplicial complex on which $\Gamma_2(n)$ acts. In Section 5, we will introduce another simply connected simplicial complex on which $\Gamma_2(n)$ acts for $n \geq 4$. Finally, in Section 6, we will obtain the presentation of Theorem 1.1, by this action and induction on n .

We now explain about an application of Theorem 1.1. For $g \geq 1$, let N_g denote a non-orientable closed surface of genus g , that is, N_g is a connected sum of g real projective planes. Let $\cdot : H_1(N_g; \mathbb{R}) \times H_1(N_g; \mathbb{R}) \rightarrow \mathbb{Z}_2$ denote the mod 2 intersection form, and let $\text{Aut}(H_1(N_g; \mathbb{R}), \cdot)$ denote the group of automorphisms over $H_1(N_g; \mathbb{R})$ preserving the mod

2010 *Mathematics Subject Classification.* 57M07, 20F05, 20F65.

Key words and phrases. congruence subgroup, presentation.

2 intersection form \cdot , where $R = \mathbb{Z}$ or \mathbb{Z}_2 . Consider the natural epimorphism

$$\Phi_g : \text{Aut}(H_1(N_g; \mathbb{Z}), \cdot) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}_2), \cdot).$$

McCarthy and Pinkall [7] showed that $\Gamma_2(g-1)$ is isomorphic to $\ker \Phi_g$.

We denote by $\mathcal{M}(N_g)$ the group of isotopy classes of diffeomorphisms over N_g . The group $\mathcal{M}(N_g)$ is called the *mapping class group* of N_g . In [7] and [4], it is shown that the natural homomorphism $\mathcal{M}(N_g) \rightarrow \text{Aut}(H_1(N_g; R), \cdot)$ is surjective, where $R = \mathbb{Z}$ or \mathbb{Z}_2 . Let $\mathcal{I}(N_g)$ denote the kernel of $\mathcal{M}(N_g) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}), \cdot)$. We say $\mathcal{I}(N_g)$ the *Torelli group* of N_g . In [5], Hirose and the author obtained a generating set of $\mathcal{I}(N_g)$ for $g \geq 4$, using Theorem 1.1.

2. PRELIMINARIES

In this section, we explain about some facts for presentations of groups.

2.1. Basics on presentations of groups.

Let G_1, G_2 and G_3 be groups with a short exact sequence

$$1 \rightarrow G_1 \xrightarrow{\phi} G_2 \xrightarrow{\pi} G_3 \rightarrow 1.$$

If G_1 and G_3 are presented then we can obtain a presentation of G_2 . In particular, if G_1 and G_3 are finitely presented then G_2 can be finitely presented.

More precisely, a presentation of G_2 is obtained as follows. Let $G_1 = \langle X_1 \mid R_1 \rangle$ and $G_3 = \langle X_3 \mid R_3 \rangle$. For each $x \in X_3$, we choose $\tilde{x} \in \pi^{-1}(x)$. We put $X_2 = \{\phi(x_1), \tilde{x}_3 \mid x_1 \in X_1, x_3 \in X_3\}$. For $r = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_k^{\varepsilon_k} \in R_3$, let $\tilde{r} = \tilde{a}_1^{\varepsilon_1} \tilde{a}_2^{\varepsilon_2} \cdots \tilde{a}_k^{\varepsilon_k}$. For $g \in \ker \pi$, let \bar{g} be a word over $\phi(X_1)$ with $g = \bar{g}$. Let $A = \{\phi(r_1) \mid r_1 \in R_1\}$, $B = \{\tilde{r}_3 \tilde{r}_3^{-1} \mid r_3 \in R_3\}$ and $C = \{\tilde{x}_3 \phi(x_1) \tilde{x}_3^{-1} \tilde{x}_3 \phi(x_1) \tilde{x}_3^{-1} \mid x_1 \in X_1, x_3 \in X_3\}$. We put $R_2 = A \cup B \cup C$. Then we have $G_2 = \langle X_2 \mid R_2 \rangle$.

In addition, if there is a homomorphism $\rho : G_3 \rightarrow G_2$ such that $\pi \circ \rho = \text{id}_{G_3}$, choose $\tilde{x} = \rho(x) \in \pi(x)^{-1}$ for $x \in X_1$. Then, we have the relation $\tilde{r} = 1$ in G_2 for $r \in R_3$.

If G_2 is presented then we can examine a presentation of G_1 , by the Reidemeister-Schreier method. In particular, if G_3 is a finite group, that is, the index of $\text{Im} \phi$ is finite, and G_2 can be finitely presented, then G_1 can be finitely presented.

For further information see [6].

2.2. Presentations of groups acting on a simplicial complex.

Let X be a simplicial complex, and let G be a group acting on X by isomorphisms as a simplicial map. We suppose that the action of G on X is *without rotation*, that is, for a simplex $\Delta \in X$ and $g \in G$, if $g(\Delta) = \Delta$ then $g(v) = v$ for all vertices $v \in \Delta$. For a simplex $\Delta \in X$, let G_Δ be the stabilizer of Δ . For $k \geq 0$, the k -skeleton $X^{(k)}$ is the subcomplex of X consisting of all simplices of dimension at most k .

Consider a homomorphism $\Phi : \bigstar_{v \in X^{(0)}} G_v \rightarrow G$. For $g \in G$, if g stabilizes a vertex $w \in X^{(0)}$, we denote g by g_w as an element in $G_w < \bigstar_{v \in X^{(0)}} G_v$. For a 1-simplex $\{v, w\} \in X$ and $g \in G_v \cap G_w$, we have $g_v g_w^{-1} \in \ker \Phi$ and call $g_v g_w^{-1}$ the *edge relator*.

At first, for any 1-simplex $\{v, w\}$, choose an orientation such that orientations are preserved by the action of G . Namely, orientations of $\{v, w\}$ and $g\{v, w\}$ are compatible for all $g \in G$. We denote the oriented 1-simplex $\{v, w\}$ by (v, w) . Similarly, choose orders of 2-simplices, and denote the ordered 2-simplex $\{v_1, v_2, v_3\}$ by (v_1, v_2, v_3) . For an oriented 1-simplex $e = (v, w)$, let $o(e) = v$ and $t(e) = w$. For an oriented 2-simplex $\tau = (v_1, v_2, v_3)$, we call v_1 the base point of τ .

Next, choose an oriented tree T of X such that a set of vertices of T is a set of representative elements for vertices of the orbit space $G \backslash X$. Let V denote the set of vertices of T . In addition, choose a set E of representative elements for oriented 1-simplices of $G \backslash X$ such that $o(e) \in V$ for $e \in E$ and 1-simplices of T is in E , and a set F of representative elements for ordered 2-simplices of $G \backslash X$ such that the base point of τ is in V for $\tau \in F$. For $e \in E$, let $w(e)$ denote the element in V which is equivalent to $t(e)$ by the action of G , and choose $g_e \in G$ such that $g_e(w(e)) = t(e)$ and $g_e = 1$ if $e \in T$.

For a 1-simplex $\{v, w\}$ with $v \in V$, note that $\{v, w\} = \{o(e), hg_e w(e)\}$ or $\{w(e), hg_e^{-1} o(e)\}$ for some $e \in E$ and $h \in G_v$. Then we define respectively $g_{\{v, w\}} = hg_e$ or hg_e^{-1} . Let α be a loop in X starting at a vertex of V . We denote $\alpha = \{v_i, \{v_i, v_{i+1}\} \mid 1 \leq i \leq k, v_{k+1} = v_1\}$. Note that $v_1, g_1^{-1} v_2 \in V$, where $g_1 = g_{\{v_1, v_2\}}$. For $2 \leq i \leq k$, define $g_i = g_{g_{i-1}^{-1} \dots g_1^{-1} \{v_i, v_{i+1}\}}$, inductively. Note that for $2 \leq i \leq k$, there exists an oriented 1-simplex e_i such that $o(e_i) \in V$ and $\{v_i, v_{i+1}\} = g_1 g_2 \dots g_{i-1} \{o(e_i), t(e_i)\}$. Let $g_\alpha = g_1 g_2 \dots g_k$. We have $g_\alpha(v_1) = v_1$, that is, $g_\alpha \in G_{v_1}$.

For $e \in E$, put a word \hat{g}_e . For a 1-simplex $\{v, w\}$ with $v \in V$, let $\hat{g}_{\{v, w\}} = h\hat{g}_e$ or $h\hat{g}_e^{-1}$ if $g_{\{v, w\}} = hg_e$ or hg_e^{-1} , respectively. For a loop α in X starting at a vertex of V , let $\hat{g}_\alpha = \hat{g}_1 \hat{g}_2 \dots \hat{g}_k$ if $g_\alpha = g_1 g_2 \dots g_k$. Note that we can define g_τ and \hat{g}_τ for $\tau \in F$, regarding τ as a loop in X . Let $\hat{G} = \left(\begin{smallmatrix} * \\ v \in V \end{smallmatrix} G_v \right) * \left(\begin{smallmatrix} * \\ e \in E \end{smallmatrix} \langle \hat{g}_e \rangle \right)$.

The following theorem is a special case of the result of Brown [1].

Theorem 2.1 ([1]). *Let X be a simply connected simplicial complex, and let G be a group acting without rotation on X by isomorphisms as a simplicial map. Then G is isomorphic to the quotient of \hat{G} by the normal subgroup generated by followings*

- (1) \hat{g}_e , where $e \in T$,
- (2) $\hat{g}_e^{-1} A_{o(e)} \hat{g}_e (g_e^{-1} A_{g_e})_{w(e)}^{-1}$, where $e \in E$ and $A \in G_e$,
- (3) $\hat{g}_\tau g_\tau^{-1}$, where $\tau \in F$.

3. PROOF OF THE CASE $n = 2$ OF THEOREM 1.1

In this section, we prove the following proposition.

Proposition 3.1. $\Gamma_2(2)$ has a finite presentation with generators E_{12} , E_{21} , F_1 and F_2 , and with relators F_1^2 , F_2^2 , $(E_{12}F_1)^2$, $(E_{12}F_2)^2$, $(E_{21}F_1)^2$, $(E_{21}F_2)^2$ and $(F_1F_2)^2$.

3.1. The Reidemeister Schreier method.

Let x, y and z be

$$x = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

At first, we prove the next lemma.

Lemma 3.2. $GL(2; \mathbb{Z})$ has a presentation with

$$GL(2; \mathbb{Z}) = \langle x, y, z \mid xyxy^{-1}x^{-1}y^{-1}, (xy)^6, z^2, xzyz \rangle.$$

Proof. In [8], it is known that $SL(2; \mathbb{Z})$ has a presentation with

$$SL(2; \mathbb{Z}) = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1}, (xy)^6 \rangle.$$

Consider the short exact sequence

$$1 \rightarrow SL(2; \mathbb{Z}) \rightarrow GL(2; \mathbb{Z}) \rightarrow \{\pm 1\} \rightarrow 1.$$

Note that $\{\pm 1\} = \langle \det z \mid (\det z)^2 \rangle$. Then we have that $GL(2; \mathbb{Z})$ has a presentation with generators x, y and z , and with the following relations

- $xyxy^{-1}x^{-1}y^{-1} = 1, (xy)^6 = 1,$
- $z^2 = 1,$
- $zxz^{-1} = y^{-1}, zyz^{-1} = x^{-1}.$

Since $z^2 = 1$, we have $zxzy = 1$ and $zyzx = 1$. Moreover the equation $zxzy = zyzx = 1$ is obtained from $xzyz = 1$. Therefore, we obtain the claim. \square

Next we consider the short exact sequence

$$1 \rightarrow \Gamma_2(2) \rightarrow GL(2; \mathbb{Z}) \xrightarrow{\pi} GL(2; \mathbb{Z}_2) \rightarrow 1.$$

For $0 \leq i \leq 5$, let $a_i \in GL(2; \mathbb{Z})$ be

$$\begin{aligned} a_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & a_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & a_2 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\ a_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & a_4 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, & a_5 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \end{aligned}$$

and let $U = \{a_0, a_1, a_2, a_3, a_4, a_5\}$. Since each of a_i is denoted by $a_0 = 1, a_1 = x^{-1}, a_2 = y, a_3 = z, a_4 = x^{-1}z$ and $a_5 = yz$, as a word over $\{x, y, z\}$, we have that U is a Schreier transversal for $\Gamma_2(2)$ in $GL(2; \mathbb{Z})$ (see [6]). For $A \in GL(2; \mathbb{Z})$, we define $\bar{A} = a_i$ if $\pi(A) = \pi(a_i)$. Let B be the set of matrices $\bar{wa_i}^{-1}wa_i$ with $wa_i \notin U$, where $0 \leq i \leq 5$ and $w = x^{\pm 1}, y^{\pm 1}$ and z . Then we have

$$B = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \right\}$$

(see Table 1). Note that B is a generating set of $\Gamma_2(2)$ (see [6]). It is clear that

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1}, \quad \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1}.$$

Thus, by Tietze transformations, we obtain the generating set $B' = \{g_1, g_2, g_3, g_4\}$ of $\Gamma_2(2)$, where

$$g_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.$$

$\bar{wa_i}^{-1}wa_i$	$w = x$	$w = x^{-1}$	$w = y$	$w = y^{-1}$	$w = z$
$i = 0$	$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$i = 1$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$i = 2$	$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$i = 3$	$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$i = 4$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$i = 5$	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

TABLE 1. The matrix $\bar{wa_i}^{-1}wa_i$.

We now prove the next lemma.

Lemma 3.3. *Let $r = r_1 r_2 \cdots r_n \in GL(2; \mathbb{Z})$. Then for $0 \leq i \leq 5$ and $1 \leq j \leq n-1$, we have*

$$\overline{r_j(r_{j+1} \cdots r_n)a_i} = \overline{(r_j r_{j+1} \cdots r_n)a_i}.$$

Proof. Note that $\overline{A} = \overline{B}$ if and only if $\pi(A) = \pi(B)$. We calculate

$$\begin{aligned} \pi(\overline{r_j(r_{j+1} \cdots r_n)a_i}) &= \pi(r_j)\pi(\overline{(r_{j+1} \cdots r_n)a_i}) \\ &= \pi(r_j)\pi((r_{j+1} \cdots r_n)a_i) \\ &= \pi((r_j r_{j+1} \cdots r_n)a_i). \end{aligned}$$

Therefore, we obtain the claim. \square

Let R be the set of relators of $GL(2; \mathbb{Z})$ in Lemma 3.2. For any $r = r_1 r_2 \cdots r_n \in R$ and $0 \leq i \leq 5$, we define a word s_{ri} over B' as follows.

$$s_{ri} = (a_i^{-1} r_1 \overline{(r_2 \cdots r_n)a_i}) (\overline{(r_2 \cdots r_n)a_i})^{-1} r_2 \overline{(r_3 \cdots r_n)a_i} \cdots (\overline{r_n a_i})^{-1} r_n a_i).$$

Let $\widehat{S} = \{s_{ri} \mid r \in R, 0 \leq i \leq 5\}$. Then \widehat{S} is a set of relators of $\Gamma_2(2)$ (see [6]). Hence we have $\Gamma_2(2) = \langle B' \mid \widehat{S} \rangle$.

3.2. Proof of Proposition 3.1.

We now write all elements in \widehat{S} as a product of elements in B' . Let $[w] = \overline{w}^{-1}w$.

For $r = xyxy^{-1}x^{-1}y^{-1}$, we have

$$\begin{aligned} s_{r0} &= [xa_1][ya_4][xa_3][y^{-1}a_5][x^{-1}a_2][y^{-1}a_0] \\ &= (g_4 g_3^{-1})^2, \\ s_{r1} &= [xa_0][ya_2][xa_5][y^{-1}a_3][x^{-1}a_4][y^{-1}a_1] \\ &= (g_1^{-1} g_3 g_4)^2, \\ s_{r2} &= [xa_5][ya_3][xa_4][y^{-1}a_1][x^{-1}a_0][y^{-1}a_2] \\ &= g_4^2, \\ s_{r3} &= [xa_4][ya_1][xa_0][y^{-1}a_2][x^{-1}a_5][y^{-1}a_3] \\ &= (g_2 g_1^{-1})^2, \\ s_{r4} &= [xa_3][ya_5][xa_2][y^{-1}a_0][x^{-1}a_1][y^{-1}a_4] \\ &= (g_3^{-1} g_1 g_2)^2, \\ s_{r5} &= [xa_2][ya_0][xa_1][y^{-1}a_4][x^{-1}a_3][y^{-1}a_5] \\ &= g_2^2. \end{aligned}$$

For $r = (xy)^6$, we have

$$\begin{aligned}
s_{r0} &= [xa_1][ya_4][xa_3][ya_5][xa_2][ya_0][xa_1][ya_4][xa_3][ya_5][xa_2][ya_0] \\
&= (g_4g_3^{-1}g_1g_2)^2, \\
s_{r1} &= [xa_0][ya_2][xa_5][ya_3][xa_4][ya_1][xa_0][ya_2][xa_5][ya_3][xa_4][ya_1] \\
&= (g_1^{-1}g_3g_4g_2)^2, \\
s_{r2} &= [xa_5][ya_3][xa_4][ya_1][xa_0][ya_2][xa_5][ya_3][xa_4][ya_1][xa_0][ya_2] \\
&= (g_4g_2g_1^{-1}g_3)^2, \\
s_{r3} &= [xa_4][ya_1][xa_0][ya_2][xa_5][ya_3][xa_4][ya_1][xa_0][ya_2][xa_5][ya_3] \\
&= (g_2g_1^{-1}g_3g_4)^2, \\
s_{r4} &= [xa_3][ya_5][xa_2][ya_0][xa_1][ya_4][xa_3][ya_5][xa_2][ya_0][xa_1][ya_4] \\
&= (g_3^{-1}g_1g_2g_4)^2, \\
s_{r5} &= [xa_2][ya_0][xa_1][ya_4][xa_3][ya_5][xa_2][ya_0][xa_1][ya_4][xa_3][ya_5] \\
&= (g_2g_4g_3^{-1}g_1)^2.
\end{aligned}$$

For $r = z^2$ and $0 \leq i \leq 5$, since $\overline{za_i}^{-1}za_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $s_{ri} = 1$. For $r = xzyz$, we have

$$\begin{aligned}
s_{r0} &= [xa_1][za_5][ya_3][za_0] = 1, \\
s_{r1} &= [xa_0][za_3][ya_5][za_1] = g_1^{-1}g_1 = 1, \\
s_{r2} &= [xa_5][za_1][ya_4][za_2] = g_4^2, \\
s_{r3} &= [xa_4][za_2][ya_0][za_3] = 1, \\
s_{r4} &= [xa_3][za_0][ya_2][za_4] = g_3^{-1}g_3 = 1, \\
s_{r5} &= [xa_2][za_4][ya_1][za_5] = g_2^2.
\end{aligned}$$

Note that $s_{(xy)^{60}} = s_{(xy)^{64}} = s_{(xy)^{65}}$, $s_{(xy)^{61}} = s_{(xy)^{62}} = s_{(xy)^{63}}$, up to conjugation, and $s_{xzyz2} = s_{xyxy^{-1}x^{-1}y^{-12}}$, $s_{xzyz5} = s_{xyxy^{-1}x^{-1}y^{-15}}$. Therefore, $\Gamma_2(2)$ has a presentation with generators g_1, g_2, g_3, g_4 and with relators $(g_4g_3^{-1})^2, (g_1^{-1}g_3g_4)^2, g_4^2, (g_2g_1^{-1})^2, (g_3^{-1}g_1g_2)^2, g_2^2, (g_4g_3^{-1}g_1g_2)^2$ and $(g_1^{-1}g_3g_4g_2)^2$.

Finally, we put $E_{12} = g_1$, $E_{21} = g_3$, $F_1 = g_4g_3^{-1}$ and $F_2 = g_2g_1^{-1}$. Note that $g_1 = E_{12}$, $g_2 = F_2E_{12}$, $g_3 = E_{21}$ and $g_4 = F_1E_{21}$. By Tietze transformations, we conclude that $\Gamma_2(2)$ has a finite presentation with generators E_{12}, E_{21}, F_1 and F_2 , and with relators $F_1^2, F_2^2, (E_{12}F_1)^2, (E_{12}F_2)^2, (E_{21}F_1)^2, (E_{21}F_2)^2$ and $(F_1F_2)^2$.

Thus, the proof of Proposition 3.1 is completed. Therefore, Theorem 1.1 is valid when $n = 2$.

4. PROOF OF THE CASE $n = 3$ OF THEOREM 1.1

In this section, we prove the following proposition.

Proposition 4.1. $\Gamma_2(3)$ has a finite presentation with generators $E_{12}, E_{13}, E_{21}, E_{23}, E_{31}, E_{32}, F_1, F_2$ and F_3 , and with the following relators

- (1) F_1^2, F_2^2, F_3^2 ,
- (2) $(E_{12}F_1)^2, (E_{12}F_2)^2, (E_{13}F_1)^2, (E_{13}F_3)^2, (E_{21}F_2)^2, (E_{21}F_1)^2, (E_{23}F_2)^2, (E_{23}F_3)^2, (E_{31}F_3)^2, (E_{31}F_1)^2, (E_{32}F_3)^2, (E_{32}F_2)^2, (F_1F_2)^2, (F_1F_3)^2, (F_2F_3)^2$,

- (3) (a) $[E_{12}, E_{13}], [E_{21}, E_{23}], [E_{31}, E_{32}], [E_{21}, E_{31}], [E_{12}, E_{32}], [E_{13}, E_{23}], [E_{12}, F_3],$
 $[E_{21}, F_3], [E_{13}, F_2], [E_{31}, F_2], [E_{23}, F_1], [E_{32}, F_1], [E_{32}, E_{13}]E_{12}^2, [E_{23}, E_{12}]E_{13}^2,$
 $[E_{31}, E_{23}]E_{21}^2, [E_{13}, E_{21}]E_{23}^2, [E_{21}, E_{32}]E_{31}^2, [E_{12}, E_{31}]E_{32}^2,$
 (b) $[E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32}, E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23}].$

4.1. Preparation.

For $R = \mathbb{Z}$ or \mathbb{Z}_2 , let $\mathcal{B}_n(R)$ denote the simplicial complex whose $(k-1)$ -simplex $\{x_1, x_2, \dots, x_k\}$ is the set of k -vectors $x_i \in R^n$ such that x_1, x_2, \dots, x_k are mutually different column vectors of a matrix $A \in GL(n; R)$. In [2], Day and Putman proved that $\mathcal{B}_n(\mathbb{Z})$ is $(n-2)$ -connected. Here, a simplicial complex X is m -connected if its geometric realization $|X|$ is m -connected. In addition, X is -1 -connected if X is nonempty. Note that there is the natural left action $\Gamma_2(n) \times \mathcal{B}_n(\mathbb{Z}) \rightarrow \mathcal{B}_n(\mathbb{Z})$ defined by $A\{x_1, x_2, \dots, x_k\} = \{Ax_1, Ax_2, \dots, Ax_k\}$ for $A \in \Gamma_2(n)$ and $\{x_1, x_2, \dots, x_k\} \in \mathcal{B}_n(\mathbb{Z})$, and that the action is without rotation.

In this section, we consider the case $n = 3$. Since $GL(3; \mathbb{Z}_2)$ is the quotient of $GL(3; \mathbb{Z})$ by $\Gamma_2(3)$, it follows that the orbit space $\Gamma_2(3) \backslash \mathcal{B}_3(\mathbb{Z})$ is isomorphic to $\mathcal{B}_3(\mathbb{Z}_2)$. Let $\varphi : \mathcal{B}_3(\mathbb{Z}) \rightarrow \mathcal{B}_3(\mathbb{Z}_2)$ be a natural surjection induced by the surjection $GL(3; \mathbb{Z}) \twoheadrightarrow GL(3; \mathbb{Z}_2)$.

For $1 \leq i \leq 7$, let v_i be $v_1 = e_1, v_2 = e_2, v_3 = e_3, v_4 = e_1 + e_2, v_5 = e_1 + e_3, v_6 = e_2 + e_3$ and $v_7 = e_1 + e_2 + e_3$, where e_1, e_2 and e_3 are canonical normal vectors in \mathbb{Z}^3 . Then, the vertices of $\mathcal{B}_3(\mathbb{Z}_2)$ are $\varphi(v_i)$, the 1-simplices are $\varphi(\{v_i, v_j\})$, and the 2-simplices are $\varphi(\{v_i, v_j, v_k\})$, where $\{i, j, k\}$ is not $\{1, 2, 4\}, \{1, 3, 5\}, \{1, 6, 7\}, \{2, 3, 6\}, \{2, 5, 7\}, \{3, 4, 7\}$ and $\{4, 5, 6\}$. (Note that $\{v_1, v_2, v_4\}, \{v_1, v_3, v_5\}, \{v_1, v_6, v_7\}, \{v_2, v_3, v_6\}, \{v_2, v_5, v_7\}, \{v_3, v_4, v_7\}$ and $\{v_4, v_5, v_6\}$ are not 2-simplices of $\mathcal{B}_3(\mathbb{Z})$.)

We prove the next lemma.

Lemma 4.2. $\Gamma_2(3)$ is isomorphic to the quotient of $\bigast_{1 \leq i \leq 7} \Gamma_2(3)_{v_i}$ by the normal subgroup generated by edge relators.

For the definition of the edge relator, see Subsection 2.2.

Proof. We set followings

- $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\},$
- $T = \{(v_1, v_i) \mid 2 \leq i \leq 7\} \cup V,$
- $E = \{(v_i, v_j) \mid 1 \leq i < j \leq 7\},$
- $F = \{(v_i, v_j, v_k) \mid 1 \leq i < j < k \leq 7, \varphi(\{v_i, v_j, v_k\}) \in \mathcal{B}_3(\mathbb{Z}_2)\}.$

For $e = (v_i, v_j) \in E$, since $w(e) = t(e)$, we choose $g_e = 1$, and write $g_{ij} = g_e$. By Theorem 2.1, $\Gamma_2(3)$ is isomorphic to the quotient of $\left(\bigast_{1 \leq i \leq 7} \Gamma_2(3)_{v_i} \right) * \left(\bigast_{1 \leq i < j \leq 7} \langle \hat{g}_{ij} \rangle \right)$ by the normal subgroup generated by followings

- (1) \hat{g}_{1i} , where $2 \leq i \leq 7$,
- (2) $\hat{g}_{ij}^{-1} X_{v_i} \hat{g}_{ij} X_{v_j}^{-1}$, where $1 \leq i < j \leq 7$ and $X \in \Gamma_2(3)_{(v_i, v_j)}$,
- (3) $\hat{g}_\tau g_\tau^{-1}$, where $\tau \in F$.

Note that $g_\tau = g_{ij} g_{jk} g_{ik}^{-1}$ for $\tau = (v_i, v_j, v_k)$. Hence, the relation $\hat{g}_\tau g_\tau^{-1} = 1$ is equivalent to the relation $\hat{g}_{ij} \hat{g}_{jk} = \hat{g}_{ik}$. Since $\hat{g}_{1i} = 1$ for $2 \leq i \leq 7$, we have the relation $\hat{g}_{ij} = 1$ for $2 \leq i < j \leq 7$ except for $(i, j) = (2, 4), (3, 5)$ and $(6, 7)$. For example, the relation $\hat{g}_{23} = 1$ is obtained from the relation $\hat{g}_{12} \hat{g}_{23} = \hat{g}_{13}$. In addition, relations $\hat{g}_{24} = 1, \hat{g}_{35} = 1$ and $\hat{g}_{67} = 1$ are obtained from relations $\hat{g}_{23} \hat{g}_{34} = \hat{g}_{24}, \hat{g}_{23} \hat{g}_{35} = \hat{g}_{25}$ and $\hat{g}_{26} \hat{g}_{67} = \hat{g}_{27}$, respectively. Hence, we have the relation $\hat{g}_{ij} = 1$ for $1 \leq i < j \leq 7$. Therefore, $\Gamma_2(3)$ is isomorphic to the quotient of $\bigast_{1 \leq i \leq 7} \Gamma_2(3)_{v_i}$ by the normal subgroup generated by $A =$

$\{X_{v_i}X_{v_j}^{-1} \mid 1 \leq i < j \leq 7, X \in \Gamma_2(3)_{(v_i, v_j)}\}$. Since A is the set of edge relators, we obtain the claim. \square

We next consider presentations of $\Gamma_2(3)_{v_i}$ for all $1 \leq i \leq 7$ and edge relators.

4.2. Presentations of $\Gamma_2(3)_{v_i}$.

Lemma 4.3. *For $1 \leq t \leq n$ there is a short exact sequence*

$$0 \rightarrow \mathbb{Z}^{n-1} \rightarrow \Gamma_2(n)_{e_t} \rightarrow \Gamma_2(n-1) \rightarrow 1.$$

Proof. We first note that $A \in \Gamma_2(n)_{e_t}$ is a matrix whose t -column vector is e_t . For \mathbb{Z}^{n-1} we give the presentation $\mathbb{Z}^{n-1} = \langle x_1, x_2, \dots, x_{n-1} \mid x_i x_j x_i^{-1} x_j^{-1} (1 \leq i < j \leq n-1) \rangle$. Let $\mathbb{Z}^{n-1} \rightarrow \Gamma_2(n)_{e_t}$ be the homomorphism which sends x_i to E_{ti} when $i < t$ and to E_{ti+1} when $i \geq t$. Let $\Gamma_2(n)_{e_t} \rightarrow \Gamma_2(n-1)$ be the homomorphism which sends A to A_{tt} , where A_{ij} is the $(n-1)$ -submatrix of A obtained by removing the i -row vector and the j -column vector of A . Then, it follows that the sequence $0 \rightarrow \mathbb{Z}^{n-1} \rightarrow \Gamma_2(n)_{e_t} \rightarrow \Gamma_2(n-1) \rightarrow 1$ is exact. \square

Remark 4.4. *Let $\rho_t : \Gamma_2(n-1) \rightarrow \Gamma_2(n)_{e_t}$ be the homomorphism defined by*

$$\begin{aligned} \rho_t(E_{ij}) &= \begin{cases} (E_{ij})_{e_t} & (\text{when } i, j \leq t-1), \\ (E_{ij+1})_{e_t} & (\text{when } i \leq t-1, j \geq t), \\ (E_{i+1j})_{e_t} & (\text{when } j \leq t-1, i \geq t), \\ (E_{i+1j+1})_{e_t} & (\text{when } i, j \geq t), \end{cases} \\ \rho_t(F_i) &= \begin{cases} (F_i)_{e_t} & (\text{when } i \leq t-1), \\ (F_{i+1})_{e_t} & (\text{when } i \geq t), \end{cases} \end{aligned}$$

where subscripts e_t are added in order to indicate that these are the elements of $\Gamma_2(n)_{e_t}$, that is, we write A_{e_t} for $A \in \Gamma_2(n)_{e_t}$. Put $\Gamma_2(n-1) = \langle X \mid Y \rangle$. Then, from Lemma 4.3, $\Gamma_2(n)_{e_t}$ is generated by

- $(E_{ti})_{e_t}$ for $1 \leq i \leq n$ with $i \neq t$,
- $(E_{ij})_{e_t}, (F_i)_{e_t}$ for $1 \leq i, j \leq n$ with $i \neq j$ and $i, j \neq t$,

and has relators

- (1) $[(E_{ti})_{e_t}, (E_{tj})_{e_t}]$ for $1 \leq i, j \leq n$ with $i \neq j$,
- (2) $\rho_t(y)$ for $y \in Y$,
- (3)
 - $(E_{ij})_{e_t}^{-1} (E_{ti})_{e_t} (E_{ij})_{e_t} \cdot (E_{tj})_{e_t}^{-2} (E_{ti})_{e_t}^{-1}$ for $1 \leq i, j \leq n$ with $i \neq j$ and $i, j \neq t$,
 - $(E_{ij})_{e_t}^{-1} (E_{tj})_{e_t} (E_{ij})_{e_t} \cdot (E_{tj})_{e_t}^{-1}$ for $1 \leq i, j \leq n$ with $i \neq j$ and $i, j \neq t$,
 - $(E_{ij})_{e_t}^{-1} (E_{tk})_{e_t} (E_{ij})_{e_t} \cdot (E_{tk})_{e_t}^{-1}$ for $1 \leq i, j, k \leq n$ with $i, j, k \neq t$ and i, j, k are mutually different (when $n \geq 4$),
 - $(F_i)_{e_t}^{-1} (E_{ti})_{e_t} (F_i)_{e_t} \cdot (E_{ti})_{e_t}$ for $1 \leq i \leq n$ with $i \neq t$,
 - $(F_i)_{e_t}^{-1} (E_{tj})_{e_t} (F_i)_{e_t} \cdot (E_{tj})_{e_t}^{-1}$ for $1 \leq i, j \leq n$ with $i \neq j$ and $i, j \neq t$.

The relators (3) can be rephrased as follows.

- $[(E_{ij})_{e_t}, (E_{ti})_{e_t}] (E_{tj})_{e_t}^2$ for $1 \leq i, j \leq n$ with $i \neq j$ and $i, j \neq t$,
- $[(E_{ij})_{e_t}, (E_{tj})_{e_t}]$ for $1 \leq i, j \leq n$ with $i \neq j$ and $i, j \neq t$,
- $[(E_{ij})_{e_t}, (E_{tk})_{e_t}]$ for $1 \leq i, j, k \leq n$ with $i, j, k \neq t$ and i, j, k are mutually different (when $n \geq 4$),
- $((E_{ti})_{e_t} (F_i)_{e_t})^2$ for $1 \leq i \leq n$ with $i \neq t$,
- $[(E_{tj})_{e_t}, (F_i)_{e_t}]$ for $1 \leq i, j \leq n$ with $i \neq j$ and $i, j \neq t$.

By Lemma 4.3, Remark 4.4 and Proposition 3.1, we have the following.

Lemma 4.5. $\Gamma_2(3)_{v_1}$ has a finite presentation with generators $(E_{12})_{v_1}$, $(E_{13})_{v_1}$, $(E_{23})_{v_1}$, $(E_{32})_{v_1}$, $(F_2)_{v_1}$ and $(F_3)_{v_1}$, and with the following relators

$$\begin{aligned} (1.1) & ((F_2)_{v_1})^2, ((F_3)_{v_1})^2, \\ (1.2) & ((E_{12})_{v_1}(F_2)_{v_1})^2, ((E_{13})_{v_1}(F_3)_{v_1})^2, ((E_{23})_{v_1}(F_2)_{v_1})^2, ((E_{23})_{v_1}(F_3)_{v_1})^2, \\ & ((E_{32})_{v_1}(F_2)_{v_1})^2, ((E_{32})_{v_1}(F_3)_{v_1})^2, ((F_2)_{v_1}(F_3)_{v_1})^2, \\ (1.3) & [(E_{12})_{v_1}, (E_{13})_{v_1}], [(E_{12})_{v_1}, (E_{32})_{v_1}], [(E_{12})_{v_1}, (F_3)_{v_1}], [(E_{13})_{v_1}, (E_{23})_{v_1}], \\ & [(E_{13})_{v_1}, (F_2)_{v_1}], [(E_{23})_{v_1}, (E_{12})_{v_1}], [(E_{13})_{v_1}^2, [(E_{32})_{v_1}, (E_{13})_{v_1}], (E_{12})_{v_1}^2]. \end{aligned}$$

For $X \in GL(n; \mathbb{Z})$, let $\Phi_X : \Gamma_2(n) \rightarrow \Gamma_2(n)$ be the homomorphism defined by $\Phi_X(A) = XAX^{-1}$. Note that this definition is well-defined, since $\Gamma_2(n)$ is a normal subgroup of $GL(n; \mathbb{Z})$. For $1 \leq i, j \leq n$ with $i \neq j$, let T_{ij} denote the matrix whose (i, j) entry is 1, diagonal entries are 1 and others are 0, and let S_i denote the matrix whose (i, i) and $(i+1, i+1)$ entries are 0, other diagonal entries are 1, $(i, i+1)$ and $(i+1, i)$ entries are 1 and others are 0. Using homomorphisms Φ_X for some $X \in GL(n; \mathbb{Z})$, we provide presentations of $\Gamma_2(n)_{v_i}$ for all $2 \leq i \leq 7$.

First, considering $\Phi_{S_1} : \Gamma_2(3)_{v_1} \rightarrow \Gamma_2(3)_{v_2}$, it follows that $\Gamma_2(3)_{v_2}$ has a finite presentation with generators $(E_{21})_{v_2}$, $(E_{23})_{v_2}$, $(E_{13})_{v_2}$, $(E_{31})_{v_2}$, $(F_1)_{v_2}$ and $(F_3)_{v_2}$, and with the following relators

$$\begin{aligned} (2.1) & ((F_1)_{v_2})^2, ((F_3)_{v_2})^2, \\ (2.2) & ((E_{21})_{v_2}(F_1)_{v_2})^2, ((E_{23})_{v_2}(F_3)_{v_2})^2, ((E_{13})_{v_2}(F_1)_{v_2})^2, ((E_{13})_{v_2}(F_3)_{v_2})^2, \\ & ((E_{31})_{v_2}(F_1)_{v_2})^2, ((E_{31})_{v_2}(F_3)_{v_2})^2, ((F_1)_{v_2}(F_3)_{v_2})^2, \\ (2.3) & [(E_{21})_{v_2}, (E_{23})_{v_2}], [(E_{21})_{v_2}, (E_{31})_{v_2}], [(E_{21})_{v_2}, (F_3)_{v_2}], [(E_{23})_{v_2}, (E_{13})_{v_2}], \\ & [(E_{23})_{v_2}, (F_1)_{v_2}], [(E_{13})_{v_2}, (E_{21})_{v_2}], [(E_{23})_{v_2}^2, [(E_{31})_{v_2}, (E_{23})_{v_2}], (E_{21})_{v_2}^2]. \end{aligned}$$

Next, considering $\Phi_{S_2S_1} : \Gamma_2(3)_{v_1} \rightarrow \Gamma_2(3)_{v_3}$, it follows that $\Gamma_2(3)_{v_3}$ has a finite presentation with generators $(E_{31})_{v_3}$, $(E_{32})_{v_3}$, $(E_{12})_{v_3}$, $(E_{21})_{v_3}$, $(F_1)_{v_3}$ and $(F_2)_{v_3}$, and with the following relators

$$\begin{aligned} (3.1) & ((F_1)_{v_3})^2, ((F_2)_{v_3})^2, \\ (3.2) & ((E_{31})_{v_3}(F_1)_{v_3})^2, ((E_{32})_{v_3}(F_2)_{v_3})^2, ((E_{12})_{v_3}(F_1)_{v_3})^2, ((E_{12})_{v_3}(F_2)_{v_3})^2, \\ & ((E_{21})_{v_3}(F_1)_{v_3})^2, ((E_{21})_{v_3}(F_2)_{v_3})^2, ((F_1)_{v_3}(F_2)_{v_3})^2, \\ (3.3) & [(E_{31})_{v_3}, (E_{32})_{v_3}], [(E_{31})_{v_3}, (E_{21})_{v_3}], [(E_{31})_{v_3}, (F_2)_{v_3}], [(E_{32})_{v_3}, (E_{12})_{v_3}], \\ & [(E_{32})_{v_3}, (F_1)_{v_3}], [(E_{12})_{v_3}, (E_{31})_{v_3}], [(E_{32})_{v_3}^2, [(E_{21})_{v_3}, (E_{32})_{v_3}], (E_{31})_{v_3}^2]. \end{aligned}$$

Next, considering $\Phi_{T_{21}} : \Gamma_2(3)_{v_1} \rightarrow \Gamma_2(3)_{v_4}$, it follows that $\Gamma_2(3)_{v_4}$ has a finite presentation with generators $(E_{21}F_2E_{12}F_1)_{v_4}$, $(E_{13}E_{23})_{v_4}$, $(E_{23})_{v_4}$, $(E_{31}^{-1}E_{32})_{v_4}$, $(E_{21}F_2)_{v_4}$ and $(F_3)_{v_4}$, and with the following relators

$$\begin{aligned} (4.1) & ((E_{21}F_2)_{v_4})^2, ((F_3)_{v_4})^2, \\ (4.2) & ((E_{21}F_2E_{12}F_1)_{v_4}(E_{21}F_2)_{v_4})^2, ((E_{13}E_{23})_{v_4}(F_3)_{v_4})^2, ((E_{23})_{v_4}(E_{21}F_2)_{v_4})^2, \\ & ((E_{23})_{v_4}(F_3)_{v_4})^2, ((E_{31}^{-1}E_{32})_{v_4}(E_{21}F_2)_{v_4})^2, ((E_{31}^{-1}E_{32})_{v_4}(F_3)_{v_4})^2, ((E_{21}F_2)_{v_4}(F_3)_{v_4})^2, \\ (4.3) & [(E_{21}F_2E_{12}F_1)_{v_4}, (E_{13}E_{23})_{v_4}], [(E_{21}F_2E_{12}F_1)_{v_4}, (E_{31}^{-1}E_{32})_{v_4}], \\ & [(E_{21}F_2E_{12}F_1)_{v_4}, (F_3)_{v_4}], [(E_{13}E_{23})_{v_4}, (E_{23})_{v_4}], [(E_{13}E_{23})_{v_4}, (E_{21}F_2)_{v_4}], \\ & [(E_{23})_{v_4}, (E_{21}F_2E_{12}F_1)_{v_4}], [(E_{13}E_{23})_{v_4}^2, [(E_{31}^{-1}E_{32})_{v_4}, (E_{13}E_{23})_{v_4}], (E_{21}F_2E_{12}F_1)_{v_4}^2]. \end{aligned}$$

Next, considering $\Phi_{S_2T_{21}} : \Gamma_2(3)_{v_1} \rightarrow \Gamma_2(3)_{v_5}$, it follows that $\Gamma_2(3)_{v_5}$ has a finite presentation with generators $(E_{31}F_3E_{13}F_1)_{v_5}$, $(E_{12}E_{32})_{v_5}$, $(E_{32})_{v_5}$, $(E_{21}^{-1}E_{23})_{v_5}$, $(E_{31}F_3)_{v_5}$ and $(F_2)_{v_5}$, and with the following relators

$$\begin{aligned} (5.1) & ((E_{31}F_3)_{v_5})^2, ((F_2)_{v_5})^2, \\ (5.2) & ((E_{31}F_3E_{13}F_1)_{v_5}(E_{31}F_3)_{v_5})^2, ((E_{12}E_{32})_{v_5}(F_2)_{v_5})^2, ((E_{32})_{v_5}(E_{31}F_3)_{v_5})^2, \\ & ((E_{32})_{v_5}(F_2)_{v_5})^2, ((E_{21}^{-1}E_{23})_{v_5}(E_{31}F_3)_{v_5})^2, ((E_{21}^{-1}E_{23})_{v_5}(F_2)_{v_5})^2, ((E_{31}F_3)_{v_5}(F_2)_{v_5})^2, \end{aligned}$$

$$(5.3) \quad [(E_{31}F_3E_{13}F_1)_{v_5}, (E_{12}E_{32})_{v_5}], [(E_{31}F_3E_{13}F_1)_{v_5}, (E_{21}^{-1}E_{23})_{v_5}], \\ [(E_{31}F_3E_{13}F_1)_{v_5}, (F_2)_{v_5}], [(E_{12}E_{32})_{v_5}, (E_{32})_{v_5}], [(E_{12}E_{32})_{v_5}, (E_{31}F_3)_{v_5}], \\ [(E_{32})_{v_5}, (E_{31}F_3E_{13}F_1)_{v_5}](E_{12}E_{32})_{v_5}^2, [(E_{21}^{-1}E_{23})_{v_5}, (E_{12}E_{32})_{v_5}](E_{31}F_3E_{13}F_1)_{v_5}^2.$$

Next, considering $\Phi_{S_1S_2T_{21}} : \Gamma_2(3)_{v_1} \rightarrow \Gamma_2(3)_{v_6}$, it follows that $\Gamma_2(3)_{v_6}$ has a finite presentation with generators $(E_{32}F_3E_{23}F_2)_{v_6}$, $(E_{21}E_{31})_{v_6}$, $(E_{31})_{v_6}$, $(E_{12}^{-1}E_{13})_{v_6}$, $(E_{32}F_3)_{v_6}$ and $(F_1)_{v_6}$, and with the following relators

$$(6.1) \quad ((E_{32}F_3)_{v_6})^2, ((F_1)_{v_6})^2, \\ (6.2) \quad ((E_{32}F_3E_{23}F_2)_{v_6}(E_{32}F_3)_{v_6})^2, ((E_{21}E_{31})_{v_6}(F_1)_{v_6})^2, ((E_{31})_{v_6}(E_{32}F_3)_{v_6})^2, \\ ((E_{31})_{v_6}(F_1)_{v_6})^2, ((E_{12}^{-1}E_{13})_{v_6}(E_{32}F_3)_{v_6})^2, ((E_{12}^{-1}E_{13})_{v_6}(F_1)_{v_6})^2, ((E_{32}F_3)_{v_6}(F_1)_{v_6})^2, \\ (6.3) \quad [(E_{32}F_3E_{23}F_2)_{v_6}, (E_{21}E_{31})_{v_6}], [(E_{32}F_3E_{23}F_2)_{v_6}, (E_{12}^{-1}E_{13})_{v_6}], \\ [(E_{32}F_3E_{23}F_2)_{v_6}, (F_1)_{v_6}], [(E_{21}E_{31})_{v_6}, (E_{31})_{v_6}], [(E_{21}E_{31})_{v_6}, (E_{32}F_3)_{v_6}], \\ [(E_{31})_{v_6}, (E_{32}F_3E_{23}F_2)_{v_6}](E_{21}E_{31})_{v_6}^2, [(E_{12}^{-1}E_{13})_{v_6}, (E_{21}E_{31})_{v_6}](E_{32}F_3E_{23}F_2)_{v_6}^2.$$

Finally, considering $\Phi_{T_{31}T_{21}} : \Gamma_2(3)_{v_1} \rightarrow \Gamma_2(3)_{v_7}$, it follows that $\Gamma_2(3)_{v_7}$ has a finite presentation with generators $(E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}$, $(E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}$, $(E_{21}^{-1}E_{23})_{v_7}$, $(E_{31}^{-1}E_{32})_{v_7}$, $(E_{21}F_2)_{v_7}$ and $(E_{31}F_3)_{v_7}$, and with the following relators

$$(7.1) \quad ((E_{21}F_2)_{v_7})^2, ((E_{31}F_3)_{v_7})^2, \\ (7.2) \quad ((E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}(E_{21}F_2)_{v_7})^2, ((E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}(E_{31}F_3)_{v_7})^2, \\ ((E_{21}^{-1}E_{23})_{v_7}(E_{21}F_2)_{v_7})^2, ((E_{21}^{-1}E_{23})_{v_7}(E_{31}F_3)_{v_7})^2, ((E_{31}^{-1}E_{32})_{v_7}(E_{21}F_2)_{v_7})^2, \\ ((E_{31}^{-1}E_{32})_{v_7}(E_{31}F_3)_{v_7})^2, ((E_{21}F_2)_{v_7}(E_{31}F_3)_{v_7})^2, \\ (7.3) \quad [(E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}, (E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}], \\ [(E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}, (E_{31}^{-1}E_{32})_{v_7}], [(E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}, (E_{31}F_3)_{v_7}], \\ [(E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}, (E_{21}^{-1}E_{23})_{v_7}], [(E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}, (E_{21}F_2)_{v_7}], \\ [(E_{21}^{-1}E_{23})_{v_7}, (E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}](E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}^2, \\ [(E_{31}^{-1}E_{32})_{v_7}, (E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}](E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}^2.$$

4.3. On edge relations.

Note that

$$\Gamma_2(3)_{(v_1, v_2)} = \Gamma_2(3)_{(v_1, v_4)} = \Gamma_2(3)_{(v_2, v_4)}, \\ \Gamma_2(3)_{(v_1, v_3)} = \Gamma_2(3)_{(v_1, v_5)} = \Gamma_2(3)_{(v_3, v_5)}, \\ \Gamma_2(3)_{(v_2, v_3)} = \Gamma_2(3)_{(v_2, v_6)} = \Gamma_2(3)_{(v_3, v_6)}, \\ \Gamma_2(3)_{(v_1, v_6)} = \Gamma_2(3)_{(v_1, v_7)} = \Gamma_2(3)_{(v_6, v_7)}, \\ \Gamma_2(3)_{(v_2, v_5)} = \Gamma_2(3)_{(v_2, v_7)} = \Gamma_2(3)_{(v_5, v_7)}, \\ \Gamma_2(3)_{(v_3, v_4)} = \Gamma_2(3)_{(v_3, v_7)} = \Gamma_2(3)_{(v_4, v_7)}.$$

It follows that $\Gamma_2(3)_{(v_1, v_2)}$, $\Gamma_2(3)_{(v_1, v_4)}$ and $\Gamma_2(3)_{(v_2, v_4)}$ are generated by

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{13})_{v_1} = (E_{13})_{v_2} = (E_{13}E_{23})_{v_4}(E_{23})_{v_4}^{-1},$
- $(E_{23})_{v_1} = (E_{23})_{v_2} = (E_{23})_{v_4},$
- $(F_3)_{v_1} = (F_3)_{v_2} = (F_3)_{v_4}.$

Next, considering $\Phi_{S_2} : \Gamma_2(3)_{(v_1, v_2)} \rightarrow \Gamma_2(3)_{(v_1, v_3)}$, it follows that $\Gamma_2(3)_{(v_1, v_3)}$, $\Gamma_2(3)_{(v_1, v_5)}$ and $\Gamma_2(3)_{(v_3, v_5)}$ are generated by

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{12})_{v_1} = (E_{12})_{v_3} = (E_{12}E_{32})_{v_5}(E_{32})_{v_5}^{-1}$,
- $(E_{32})_{v_1} = (E_{32})_{v_3} = (E_{32})_{v_5}$,
- $(F_2)_{v_1} = (F_2)_{v_3} = (F_2)_{v_5}$.

Next, considering $\Phi_{S_1S_2} : \Gamma_2(3)_{(v_1, v_2)} \rightarrow \Gamma_2(3)_{(v_2, v_3)}$, it follows that $\Gamma_2(3)_{(v_2, v_3)}$, $\Gamma_2(3)_{(v_2, v_6)}$ and $\Gamma_2(3)_{(v_3, v_6)}$ are generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{21})_{v_2} = (E_{21})_{v_3} = (E_{21}E_{31})_{v_6}(E_{31})_{v_6}^{-1}$,
- $(E_{31})_{v_2} = (E_{31})_{v_3} = (E_{31})_{v_6}$,
- $(F_1)_{v_2} = (F_1)_{v_3} = (F_1)_{v_6}$.

Next, considering $\Phi_{T_{32}} : \Gamma_2(3)_{(v_1, v_2)} \rightarrow \Gamma_2(3)_{(v_1, v_6)}$, it follows that $\Gamma_2(3)_{(v_1, v_6)}$, $\Gamma_2(3)_{(v_1, v_7)}$ and $\Gamma_2(3)_{(v_6, v_7)}$ are generated by

$$\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{12})_{v_1}^{-1}(E_{13})_{v_1} = (E_{12}^{-1}E_{13})_{v_6}$
 $= (E_{31}F_3)_{v_7}(E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}(E_{21}^{-1}E_{23})_{v_7}^{-1}(E_{21}F_2)_{v_7}$
 $\cdot (E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}(E_{31}^{-1}E_{32})_{v_7}^{-1}$,
- $(E_{32})_{v_1}(F_3)_{v_1}(E_{23})_{v_1}(F_2)_{v_1} = (E_{32}F_3E_{23}F_2)_{v_6}$
 $= (E_{31}^{-1}E_{32})_{v_7}(E_{31}F_3)_{v_7}(E_{21}^{-1}E_{23})_{v_7}(E_{21}F_2)_{v_7}$,
- $(E_{32})_{v_1}(F_3)_{v_1} = (E_{32}F_3)_{v_6} = (E_{31}^{-1}E_{32})_{v_7}(E_{31}F_3)_{v_7}$.

Next, considering $\Phi_{S_1T_{32}} : \Gamma_2(3)_{(v_1, v_2)} \rightarrow \Gamma_2(3)_{(v_2, v_5)}$, it follows that $\Gamma_2(3)_{(v_2, v_5)}$, $\Gamma_2(3)_{(v_2, v_7)}$ and $\Gamma_2(3)_{(v_5, v_6)}$ are generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ -2 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{21})_{v_2}^{-1}(E_{23})_{v_2} = (E_{21}^{-1}E_{23})_{v_5} = (E_{21}^{-1}E_{23})_{v_7}$,
- $(E_{31})_{v_2}(F_3)_{v_2}(E_{13})_{v_2}(F_1)_{v_2} = (E_{31}F_3E_{13}F_1)_{v_5}$
 $= (E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7}(E_{21}^{-1}E_{23})_{v_7}^{-1}$,
- $(E_{31})_{v_2}(F_3)_{v_2} = (E_{31}F_3)_{v_5} = (E_{31}F_3)_{v_7}$.

Next, considering $\Phi_{S_2S_1T_{32}} : \Gamma_2(3)_{(v_1,v_2)} \rightarrow \Gamma_2(3)_{(v_3,v_4)}$, it follows that $\Gamma_2(3)_{(v_3,v_4)}$, $\Gamma_2(3)_{(v_3,v_7)}$ and $\Gamma_2(3)_{(v_4,v_7)}$ are generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{31})_{v_3}^{-1}(E_{32})_{v_3} = (E_{31}^{-1}E_{32})_{v_4} = (E_{31}^{-1}E_{32})_{v_7},$
- $(E_{21})_{v_3}(F_2)_{v_3}(E_{12})_{v_3}(F_1)_{v_3} = (E_{21}F_2E_{12}F_1)_{v_4}$
 $= (E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7}(E_{31}^{-1}E_{32})_{v_7}^{-1},$
- $(E_{21})_{v_3}(F_2)_{v_3} = (E_{21}F_2)_{v_4} = (E_{21}F_2)_{v_7}.$

Next, considering $\Phi_{T_{31}T_{32}} : \Gamma_2(3)_{(v_1,v_2)} \rightarrow \Gamma_2(3)_{(v_5,v_6)}$, it follows that $\Gamma_2(3)_{(v_5,v_6)}$ is generated by

$$\begin{pmatrix} -1 & -2 & 2 \\ 0 & 1 & 0 \\ -2 & -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{12}E_{32})_{v_5}^{-1}(E_{31}F_3E_{13}F_1)_{v_5} = (E_{31})_{v_6}(F_1)_{v_6}(E_{32}F_3)_{v_6}(E_{12}^{-1}E_{13})_{v_6}^{-1},$
- $(E_{32})_{v_5}(F_2)_{v_5}(E_{31}F_3)_{v_5}(E_{21}^{-1}E_{23})_{v_5}^{-1} = (E_{21}E_{31})_{v_6}^{-1}(E_{32}F_3E_{23}F_2)_{v_6},$
- $(E_{32})_{v_5}(E_{31}F_3)_{v_5} = (E_{31})_{v_6}(E_{32}F_3)_{v_6}.$

Next, considering $\Phi_{S_2T_{31}T_{32}} : \Gamma_2(3)_{(v_1,v_2)} \rightarrow \Gamma_2(3)_{(v_4,v_6)}$, it follows that $\Gamma_2(3)_{(v_4,v_6)}$ is generated by

$$\begin{pmatrix} -1 & 2 & -2 \\ -2 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & -2 \\ -2 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{13}E_{23})_{v_4}^{-1}(E_{21}F_2E_{12}F_1)_{v_4}$
 $= (E_{21}E_{31})_{v_6}(E_{31})_{v_6}^{-1}(F_1)_{v_6}(E_{32}F_3)_{v_6}(E_{32}F_3E_{23}F_2)_{v_6}(E_{12}^{-1}E_{13})_{v_6},$
- $(E_{23})_{v_4}(F_3)_{v_4}(E_{21}F_2)_{v_4}(E_{31}^{-1}E_{32})_{v_4}^{-1} = (E_{21}E_{31})_{v_6}^{-1}(E_{32}F_3E_{23}F_2)_{v_6}^{-1},$
- $(E_{23})_{v_4}(E_{21}F_2)_{v_4} = (E_{21}E_{31})_{v_6}(E_{31})_{v_6}^{-1}(E_{32}F_3)_{v_6}(E_{32}F_3E_{23}F_2)_{v_6}.$

Finally, considering $\Phi_{S_1S_2T_{31}T_{32}} : \Gamma_2(3)_{(v_1,v_2)} \rightarrow \Gamma_2(3)_{(v_4,v_5)}$, it follows that $\Gamma_2(3)_{(v_4,v_5)}$ is generated by

$$\begin{pmatrix} 3 & -2 & -2 \\ 2 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -2 & -2 \\ 0 & 1 & 0 \\ 2 & -2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following edge relations

- $(E_{13}E_{23})_{v_4}^{-1}(E_{21}F_2E_{12}F_1)_{v_4}^{-1}$
 $= (E_{12}E_{32})_{v_5}(E_{32})_{v_5}^{-1}(F_2)_{v_5}(E_{31}F_3)_{v_5}(E_{31}F_3E_{13}F_1)_{v_5}(E_{21}^{-1}E_{23})_{v_5},$
- $(E_{13}E_{23})_{v_4}(E_{23})_{v_4}^{-1}(F_3)_{v_4}(E_{21}F_2)_{v_4}(E_{21}F_2E_{12}F_1)_{v_4}(E_{31}^{-1}E_{32})_{v_4}$
 $= (E_{12}E_{32})_{v_5}^{-1}(E_{31}F_3E_{13}F_1)_{v_5}^{-1},$
- $(E_{13}E_{23})_{v_4}(E_{23})_{v_4}^{-1}(E_{21}F_2)_{v_4}(E_{21}F_2E_{12}F_1)_{v_4}$
 $= (E_{12}E_{32})_{v_5}(E_{32})_{v_5}^{-1}(E_{31}F_3)_{v_5}(E_{31}F_3E_{13}F_1)_{v_5}.$

Therefore, using Tietze transformations, by Lemma 4.2, we obtain the presentation for Proposition 4.1 (For more details see Appendix A). Thus, Theorem 1.1 is valid when $n = 3$.

5. A SIMPLICIAL COMPLEX ON WHICH $\Gamma_2(n)$ ACTS

Let $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ denote the subcomplex of $\mathcal{B}_n(\mathbb{Z})$ whose $(k-1)$ -simplex $\{x_1, x_2, \dots, x_k\}$ is the set of k -vectors $x_i \in \mathbb{Z}^n$ such that x_1, x_2, \dots, x_k are mutually different column vectors of a matrix $A \in \Gamma_2(n)$. Note that for a vertex v , we have $v \equiv e_i \pmod{2}$ for some $1 \leq i \leq n$, where e_1, e_2, \dots, e_n are canonical normal vectors in \mathbb{Z}^n . For a $(k-1)$ -simplex $\Delta = \{x_1, x_2, \dots, x_k\}$, $A \in \Gamma_2(n)$ is an *extension* of Δ if each x_i is a column vector of A .

In this section, we prove the following proposition.

Proposition 5.1. *For $n \geq 4$, the simplicial complex $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is simply connected.*

In a proof of this proposition, we will use the idea of Day-Putman [2] for proving that $\mathcal{B}_n(\mathbb{Z})$ is $(n-2)$ -connected.

5.1. Preparation.

Let X be a simplicial complex. Then we define followings.

- For a simplex $\Delta \in X$, $\text{star}_X(\Delta)$ is the subcomplex of X whose simplex $\Delta' \in X$ satisfies that $\Delta, \Delta' \subset \Delta''$ for some simplex $\Delta'' \in X$. We also define $\text{star}_X(\emptyset) = X$.
- For a simplex $\Delta \in X$, $\text{link}_X(\Delta)$ is the subcomplex of $\text{star}_X(\Delta)$ whose simplex $\Delta' \in \text{star}_X(\Delta)$ does not intersect Δ . We also define $\text{link}_X(\emptyset) = X$.

Here, we prove followings.

Lemma 5.2. *For $n \geq 2$, $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is path connected.*

Proof. We first consider the case $n = 2$. Let $v_0 = v_{01}e_1 + v_{02}e_2 \in \Gamma_2\mathcal{B}_2(\mathbb{Z})$ be a vertex. Then there exists a vertex $v_1 = v_{11}e_1 + v_{12}e_2 \in \Gamma_2\mathcal{B}_2(\mathbb{Z})$ such that $\{v_0, v_1\} \in \Gamma_2\mathcal{B}_2(\mathbb{Z})$. Note that $v_{01}v_{12} - v_{02}v_{11} = \pm 1$. By Euclidean algorithm, we can suppose that $|v_{01}| > |v_{11}|$. Similarly, there exist vertices $v_2 = v_{21}e_1 + v_{22}e_2, \dots, v_k = v_{k1}e_1 + v_{k2}e_2 \in \Gamma_2\mathcal{B}_2(\mathbb{Z})$ such that $\{v_i, v_{i+1}\} \in \Gamma_2\mathcal{B}_2(\mathbb{Z})$, $|v_{i1}| > |v_{i+1,1}|$ for $1 \leq i \leq k-1$ and $v_k = e_1$ or e_2 , for some positive integer k . Hence, $\Gamma_2\mathcal{B}_2(\mathbb{Z})$ is path connected.

Next, we suppose $n \geq 3$. Let $v, w \in \Gamma_2\mathcal{B}_n(\mathbb{Z})$ be vertices. Without loss of generality, we suppose $v \equiv e_1 \pmod{2}$ and $w \equiv e_2 \pmod{2}$. Then there is an extension $A \in \Gamma_2(n)$ of v . We write $A^{-1}w = \sum_{i=1}^n a_i e_i$. Let $S_{A^{-1}w} = \sum_{i=3}^n |a_i|$. For $3 \leq i \leq n$, if $|a_2| < |a_i|$, there is an integer $u \in \mathbb{Z}$ such that $|a_2| > |a_i + 2ua_2|$. Then we have that $S_{E_{i2}^u A^{-1}w} < S_{A^{-1}w}$ and $E_{i2}^u A^{-1}v = e_1$. If $|a_2| > |a_i| \neq 0$, there is an integer $u' \in \mathbb{Z}$ such that $|a_2 + 2u'a_i| < |a_i|$. In addition, there is an integer $u'' \in \mathbb{Z}$ such that $|a_2 + 2u''a_1| > |a_i + 2u''(a_2 + 2u'a_1)|$. Then we have that $S_{E_{i2}^{u''} E_{2i}^{u'} A^{-1}w} < S_{A^{-1}w}$ and $E_{i2}^{u''} E_{2i}^{u'} A^{-1}v = e_1$. Repeating this operation, we conclude that there exists $B \in \Gamma_2(n)$ such that $S_{Bw} = 0$ and $Bv = e_1$. Note that Bw can be regarded as a vertex in $\Gamma_2\mathcal{B}_2(\mathbb{Z})$. Hence, Bw is joined to e_1 , that is, Bw is joined to Bv . The action of B^{-1} brings the path joining Bw with Bv to the path joining w with v . Thus, $\Gamma_2\mathcal{B}_n(\mathbb{Z})$ is path connected. \square

Lemma 5.3. *Let $\Delta \in \Gamma_2\mathcal{B}_n(\mathbb{Z})$ be a $(k-1)$ -simplex. Then we have followings.*

- $\text{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)$ is isomorphic to $\text{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_1, e_2, \dots, e_k\})$ as a simplicial complex.
- $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)$ is isomorphic to $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_1, e_2, \dots, e_k\})$ as a simplicial complex.

Proof. For $\Delta = \{x_1, x_2, \dots, x_k\}$, suppose $x_j \equiv e_{i(j)} \pmod{2}$. Let $A \in \Gamma_2(n)$ be an extension of Δ . Then restrictions of the action of A^{-1} on $\Gamma_2\mathcal{B}_n(\mathbb{Z})$

$$\begin{aligned} A^{-1}|_{\text{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)} : \text{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta) &\rightarrow \text{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_{i(1)}, e_{i(2)}, \dots, e_{i(k)}\}), \\ A^{-1}|_{\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)} : \text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta) &\rightarrow \text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_{i(1)}, e_{i(2)}, \dots, e_{i(k)}\}) \end{aligned}$$

are isomorphisms as a simplicial map. It is clear that $\text{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_{i(1)}, e_{i(2)}, \dots, e_{i(k)}\})$ and $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_{i(1)}, e_{i(2)}, \dots, e_{i(k)}\})$ are respectively isomorphic to $\text{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_1, e_2, \dots, e_k\})$ and $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_1, e_2, \dots, e_k\})$. Thus, we obtain the claim. \square

Corollary 5.4. *Let $\Delta \in \Gamma_2\mathcal{B}_n(\mathbb{Z})$ be a $(k-1)$ -simplex. If $n-k \geq 2$, then $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)$ is path connected.*

Proof. By an argument similar to the proof of Lemma 5.2, we have that $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\{e_1, e_2, \dots, e_k\})$ is path connected. By Lemma 5.3, $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(\Delta)$ is also path connected. \square

5.2. Proof of Proposition 5.1.

We suppose $n \geq 4$. Let $\alpha = \{x_i, \{x_i, x_{i+1}\} \mid 1 \leq i \leq k, x_{k+1} = x_1\}$ be a loop on $\Gamma_2\mathcal{B}_n(\mathbb{Z})$. We show that α is null-homotopic.

For $v = \sum_{i=1}^n v_i e_i \in \mathbb{Z}^n$, we define $\text{Rank}(v) = |v_n|$. Let $R_\alpha = \max \text{Rank}(x_i)$.

We first prove the next lemma.

Lemma 5.5. *For a 1-simplex $\{v, w\} \in \Gamma_2\mathcal{B}_n(\mathbb{Z})$ with $\text{Rank}(v) = \text{Rank}(w) = 0$, we have $\{v, w\} \in \text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$.*

Proof. Note that $v \not\equiv w \pmod{2}$. Suppose that $v \equiv e_i, w \equiv e_j \pmod{2}$ and $i < j$. Since $\text{Rank}(v) = \text{Rank}(w) = 0$, we have that $v, w \not\equiv e_n \pmod{2}$. There exists an extension $A = (a_1 a_2 \dots a_n) \in \Gamma_2(n)$ of $\{v, w\}$. Let $S_A = \sum_{l=1}^n \text{Rank}(a_l)$. Note that S_A is odd.

First, we consider the case $S_A = 1$. Note that $\text{Rank}(a_l) = 0$ for $1 \leq l \leq n-1$ and $\text{Rank}(a_n) = 1$. Put $a_n = \sum_{i=1}^{n-1} 2b_i e_i + \varepsilon e_n$, where $\varepsilon = \pm 1$. Let $B = E_{1n}^{b_1} E_{2n}^{b_2} \dots E_{n-1n}^{b_{n-1}} F_n^{\frac{\varepsilon-1}{2}}$. Then we have $BA = (a_1 \dots a_{n-1} e_n)$. Hence, we have that $\{v, w\} = \{a_i, a_j\} \in \text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$.

Next, we suppose $S_A \geq 3$. Note that there exists $1 \leq l \leq n-1$ with $l \neq i, j$ such that $\text{Rank}(a_l) \neq 0$. If $\text{Rank}(a_l) > \text{Rank}(a_n)$, there exists an integer $u \in \mathbb{Z}$ such that $\text{Rank}(a_l + 2ua_n) < \text{Rank}(a_n)$. Then we have that AE_{nl}^u is an extension of $\{v, w\}$ and that $S_{AE_{nl}^u} < S_A$. Similarly, if $\text{Rank}(a_l) < \text{Rank}(a_n)$, there exists an integer $u' \in \mathbb{Z}$ such that $\text{Rank}(a_l) > \text{Rank}(a_n + 2u'a_l)$. Then we have that $AE_{ln}^{u'}$ is an extension of $\{v, w\}$ and that $S_{AE_{ln}^{u'}} < S_A$. Repeating this operation, we conclude that there exists an extension $A' \in \Gamma_2(n)$ of $\{v, w\}$ such that $S_{A'} = 1$. Therefore, we have $\{v, w\} \in \text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$. Thus, we obtain the claim. \square

When $R_\alpha = 0$, by this lemma, we have $\{x_i, x_{i+1}\} \in \text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$. Namely, the loop α is in $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$. Since $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$ is the subcomplex of $\text{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$ and $\text{star}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(e_n)$ is contractible, α is null-homotopic. Therefore, we next assume $R_\alpha > 0$.

Suppose that R_α is odd. There exists $1 \leq i \leq k$ such that $\text{Rank}(x_i) = R_\alpha$. Since R_α is odd, we have that $x_i \equiv e_n, x_{i\pm 1} \not\equiv e_n \pmod{2}$ and $\text{Rank}(x_{i\pm 1}) < R_\alpha$. By Corollary 5.4, we have that $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(x_i)$ is path connected. Since $x_{i\pm 1} \in \text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(x_i)$, there exists a path $\{y_j, y_l, \{y_j, y_{j+1}\} \mid 1 \leq j \leq l-1\}$ on $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(x_i)$ between x_{i-1} and x_{i+1} such that $y_1 = x_{i-1}$ and $y_l = x_{i+1}$ (see Figure 1). Since R_α is odd and $\text{Rank}(y_j)$ is even for each y_j , there exists an integer $s_j \in \mathbb{Z}$ such that $\text{Rank}(y'_j) < R_\alpha$, where $y'_j = y_j + 2s_j x_i$. We choose $s_j = 0$ if $\text{Rank}(y_j) < R_\alpha$. When $y_j \equiv e_t, y_{j+1} \equiv e_u \pmod{2}$, for an extension $A \in \Gamma_2(n)$ of $\{x_i, y_j, y_{j+1}\}$, we have that $\{x_i, y'_j, y'_{j+1}\} = \{AE_{nt}^{s_j} E_{nu}^{s_{j+1}} e_n, AE_{nt}^{s_j} E_{nu}^{s_{j+1}} e_t, AE_{nt}^{s_j} E_{nu}^{s_{j+1}} e_u\}$. Hence $\{x_i, y'_j, y'_{j+1}\}$ is a 2-simplex which has an extension $AE_{nt}^{s_j} E_{nu}^{s_{j+1}}$. Therefore we have that the path $\{y'_j, y'_l, \{y'_j, y'_{j+1}\} \mid 1 \leq j \leq l-1\}$ between x_{i-1} and x_{i+1} is in $\text{link}_{\Gamma_2\mathcal{B}_n(\mathbb{Z})}(x_i)$ (see Figure 1). Let $\alpha' = \alpha \cup \{y'_j, y'_l, \{y'_j, y'_{j+1}\} \mid 1 \leq j \leq l-1\} \setminus \{x_i, \{x_i, x_{i\pm 1}\}\}$. Then α' is homotopic to α (see Figure 1). For all x_i with $\text{Rank}(x_i) = R_\alpha$, applying the same

operation, we conclude that $R_\beta < R_\alpha$, where β is a resulting loop which is homotopic to α .

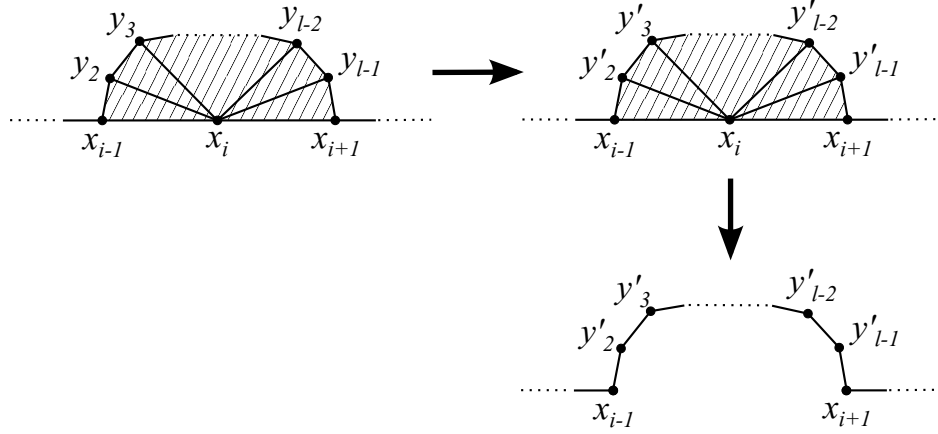


FIGURE 1. The case R_α is odd.

Next, suppose that R_α is even. There exists $1 \leq i \leq k$ such that $\text{Rank}(x_i) = R_\alpha$. Since R_α is even, we have $x_i \not\equiv e_n \pmod{2}$.

Remark 5.6. Under the assumption $n \geq 4$, we may suppose that α satisfies all of the following conditions.

- $\text{Rank}(x_{i\pm 1}) < R_\alpha$,
- $x_{i\pm 1} \not\equiv e_n \pmod{2}$,
- $x_{i-1} \not\equiv x_{i+1} \pmod{2}$.

Proof. Without loss of generality, we suppose that $x_i \equiv e_1 \pmod{2}$.

- Suppose that $\text{Rank}(x_{i-1}) = R_\alpha$. Since R_α is even we have $x_{i-1} \not\equiv e_n \pmod{2}$. Without loss of generality, we suppose that $x_{i-1} \equiv e_2 \pmod{2}$. There exists an extension $A \in \Gamma_2(n)$ of $\{x_i, x_{i-1}\}$ such that $\text{Rank}(Ae_n) < R_\alpha$. In fact, if $\text{Rank}(Ae_n) > R_\alpha$, there is an integer $u \in \mathbb{Z}$ such that $\text{Rank}(AE_{1n}^u e_n) < R_\alpha$. Then we choose AE_{1n}^u in place of A as an extension of $\{x_i, x_{i-1}\}$. (Note that $\text{Rank}(Ae_n)$ and $\text{Rank}(AE_{1n}^u e_n)$ are not equal to R_α , since these are odd.) Let $y = Ae_n$, and let $\alpha' = \alpha \cup \{y, \{x_{i-1}, y\}, \{y, x_i\}\} \setminus \{\{x_{i-1}, x_i\}\}$. Then α' is homotopic to α . Hence, considering α' in place of α , we may suppose $\text{Rank}(x_{i-1}) < R_\alpha$. Similarly, we may suppose $\text{Rank}(x_{i+1}) < R_\alpha$.
- Suppose that $x_{i-1} \equiv e_n \pmod{2}$. Since $\text{Rank}(x_{i-1})$ is odd we have $\text{Rank}(x_{i-1}) < R_\alpha$. There exists an extension $A \in \Gamma_2(n)$ of $\{x_i, x_{i-1}\}$ such that $\text{Rank}(Ae_2) < \text{Rank}(x_{i-1}) (< R_\alpha)$. In fact, if $\text{Rank}(Ae_2) > \text{Rank}(x_{i-1})$, there is an integer $u \in \mathbb{Z}$ such that $\text{Rank}(AE_{n2}^u e_2) < \text{Rank}(x_{i-1})$. Then we choose AE_{n2}^u in place of A as an extension of $\{x_i, x_{i-1}\}$. (Note that $\text{Rank}(Ae_2)$ and $\text{Rank}(AE_{n2}^u e_2)$ are not equal to $\text{Rank}(x_{i-1})$, since these are even.) Let $y = Ae_2$, and let $\alpha' = \alpha \cup \{y, \{x_{i-1}, y\}, \{y, x_i\}\} \setminus \{\{x_{i-1}, x_i\}\}$. Then α' is homotopic to α . Hence, considering α' in place of α , we may suppose $\text{Rank}(x_{i-1}) < R_\alpha$ and $x_{i-1} \not\equiv e_n \pmod{2}$. Similarly, we may suppose $\text{Rank}(x_{i+1}) < R_\alpha$ and $x_{i+1} \not\equiv e_n \pmod{2}$.
- Suppose that $\text{Rank}(x_{i\pm 1}) < R_\alpha$, $x_{i\pm 1} \not\equiv e_n \pmod{2}$ and $x_{i-1} \equiv x_{i+1} \pmod{2}$. Without loss of generality, we suppose that $x_{i\pm 1} \equiv e_2 \pmod{2}$. There exists an extension $A \in \Gamma_2(n)$ of $\{x_i, x_{i-1}\}$ such that $\text{Rank}(Ae_3) \leq \text{Rank}(x_{i-1}) (< R_\alpha)$.

In fact, if $\text{Rank}(Ae_3) > \text{Rank}(x_{i-1})$, there is an integer $u \in \mathbb{Z}$ such that $\text{Rank}(AE_{23}^u e_3) \leq \text{Rank}(x_{i-1})$. Then we choose AE_{23}^u in place of A as an extension of $\{x_i, x_{i-1}\}$. (Since $Ae_3 \not\equiv x_i, x_{i\pm 1}, e_n \pmod{2}$, we need the assumption $n \geq 4$.) Let $y = Ae_3$, and let $\alpha' = \alpha \cup \{y, \{x_{i-1}, y\}, \{y, x_i\}\} \setminus \{\{x_{i-1}, x_i\}\}$. Then α' is homotopic to α . Hence, considering α' in place of α , we may suppose that $\text{Rank}(x_{i\pm 1}) < R_\alpha$, $x_{i\pm 1} \not\equiv e_n \pmod{2}$ and $x_{i-1} \not\equiv x_{i+1} \pmod{2}$.

□

We now suppose that α satisfies the conditions of the above remark. Suppose that $x_i \equiv e_s$, $x_{i-1} \equiv e_t$ and $x_{i+1} \equiv e_u \pmod{2}$, where s, t and u are mutually different and not equal to n . Since $\{x_{i-1}, x_i\}$ is a 1-simplex in $\Gamma_2 \mathcal{B}_n(\mathbb{Z})$, there is an extension $B \in \Gamma_2(n)$ of $\{x_{i-1}, x_i\}$. We write $B^{-1}x_{i+1} = \sum_{j=1}^n a_j e_j$. It follows that there exist an even integer b_u and an odd integer b_n such that $a_u b_n - a_n b_u = \gcd(a_u, a_n)$. Then we have that

$$\begin{pmatrix} a_u/\gcd(a_u, a_n) & b_u \\ a_n/\gcd(a_u, a_n) & b_n \end{pmatrix}^{-1} \begin{pmatrix} a_u \\ a_n \end{pmatrix} = \begin{pmatrix} \gcd(a_u, a_n) \\ 0 \end{pmatrix}.$$

Let $C \in \Gamma_2(n)$ be the matrix whose (u, u) entry is $a_u/\gcd(a_u, a_n)$, (n, u) entry is $a_n/\gcd(a_u, a_n)$, (u, n) entry is b_u , (n, n) entry is b_n , other diagonal entries are 1 and other entries are 0. Then if we set $A = C^{-1}B^{-1}$, it follows that $Ax_i = e_s$, $Ax_{i-1} = e_t$ and $\text{Rank}(Ax_{i+1}) = 0$.

Since $\{e_s, Ax_{i+1}\}$ is a 1-simplex and $\text{Rank}(e_s) = \text{Rank}(Ax_{i+1}) = 0$, by Lemma 5.5, we have that $\{e_s, Ax_{i+1}\} \in \text{link}_{\Gamma_2 \mathcal{B}_n(\mathbb{Z})}(e_n)$. Therefore, we have that $e_n \in \text{link}_{\Gamma_2 \mathcal{B}_n(\mathbb{Z})}(\{e_s, Ax_{i+1}\})$. In addition, it is clear that $e_n \in \text{link}_{\Gamma_2 \mathcal{B}_n(\mathbb{Z})}(\{e_s, e_t\})$. Hence, we have that $A^{-1}e_n \in \text{link}_{\Gamma_2 \mathcal{B}_n(\mathbb{Z})}(\{x_i, x_{i\pm 1}\})$ (see Figure 2). Then, there exists an integer l such that $\text{Rank}(x'_i) < R_\alpha$, where $x'_i = A^{-1}e_n + 2lx_i$. We have also that $x'_i \in \text{link}_{\Gamma_2 \mathcal{B}_n(\mathbb{Z})}(\{x_i, x_{i\pm 1}\})$ (see Figure 2). Let $\alpha' = \alpha \cup \{\{x'_i\}, \{x'_i, x_{i\pm 1}\}\} \setminus \{x_i, \{x_i, x_{i\pm 1}\}\}$. Then α' is homotopic to α (see Figure 2). Similar to the case R_α is odd, for all x_i with $\text{Rank}(x_i) = R_\alpha$, applying the same operation, we conclude that $R_\beta < R_\alpha$, where β is a resulting loop which is homotopic to α .

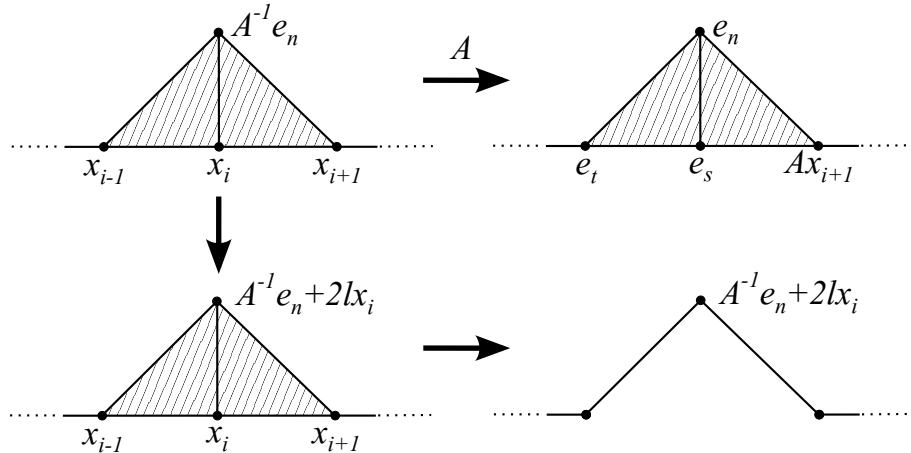


FIGURE 2. The case R_α is even.

Repeating this operation until $R_\alpha = 0$, we conclude that the loop α on $\Gamma_2 \mathcal{B}_n(\mathbb{Z})$ is null homotopic. Thus, $\Gamma_2 \mathcal{B}_n(\mathbb{Z})$ is simply connected.

6. PROOF OF THEOREM 1.1

We first prove the next proposition.

Lemma 6.1. *For any $n \geq 4$, $\Gamma_2(n)$ is isomorphic to the quotient of $\bigstar_{1 \leq i \leq n} \Gamma_2(n)_{e_i}$ by the normal subgroup generated by edge relators.*

Proof. For a $(k-1)$ -simplex $\Delta = \{x_1, x_2, \dots, x_k\} \in \Gamma_2 \mathcal{B}_n(\mathbb{Z})$ with $x_j \equiv e_{i(j)} \pmod{2}$, let $A \in \Gamma_2(n)$ be an extension of Δ . Then we have $A^{-1} \cdot \Delta = \{e_{i(1)}, e_{i(2)}, \dots, e_{i(k)}\}$. Therefore, we have

$$\Gamma_2(n) \backslash \Gamma_2 \mathcal{B}_n(\mathbb{Z}) = \{\{e_{i(1)}, e_{i(2)}, \dots, e_{i(k)}\} \mid 1 \leq k \leq n, 1 \leq i(1) < i(2) < \dots < i(k) \leq n\}.$$

It is clear that $\Gamma_2(n) \backslash \Gamma_2 \mathcal{B}_n(\mathbb{Z})$ is contractible. Note that the action of $\Gamma_2(n)$ on $\Gamma_2 \mathcal{B}_n(\mathbb{Z})$ is without rotation.

We first set followings.

- $T = \{(e_1, e_i) \mid 2 \leq i \leq n\}$.
- $E = \{(e_i, e_j) \mid 1 \leq i < j \leq n\}$.
- $F = \{(e_i, e_j, e_k) \mid 1 \leq i < j < k \leq n\}$.
- For $e \in E$, we choose $g_e = 1$, and write $g_e = g_{ij}$ when $e = (e_i, e_j)$.
- For $\tau = (e_i, e_j, e_k) \in F$, let $g_\tau = g_{ij}g_{jk}g_{ik}^{-1}$.

Then, since $\Gamma_2 \mathcal{B}_n(\mathbb{Z})$ is simply connected, it follows from Theorem 2.1 that $\Gamma_2(n)$ is isomorphic to the quotient of $\left(\bigstar_{1 \leq i \leq n} \Gamma_2(n)_{e_i}\right) * \left(\bigstar_{1 \leq i < j \leq n} \langle \hat{g}_{ij} \rangle\right)$ by the normal subgroup generated by followings

- (1) \hat{g}_{1i} , where $2 \leq i \leq n$,
- (2) $\hat{g}_{ij}^{-1} X_{e_i} \hat{g}_{ij} X_{e_j}^{-1}$, where $1 \leq i < j \leq n$ and $X \in \Gamma_2(n)_{(e_i, e_j)}$,
- (3) $\hat{g}_\tau g_\tau^{-1}$, where $\tau \in F$.

Since $g_\tau = 1$, the relation $\hat{g}_\tau g_\tau^{-1}$ is equivalent to the relation $\hat{g}_{ij} \hat{g}_{jk} = \hat{g}_{ik}$ if $\tau = (e_i, e_j, e_k)$. By relations $\hat{g}_{1i} = 1$, we have the relation $\hat{g}_{ij} = 1$ for $1 \leq i < j \leq n$. Thus, we obtain the claim. \square

Note that for $e = (e_s, e_t)$, $\Gamma_2(n)_e$ is generated by $(E_{ij})_e$ and $(F_j)_e$ for $1 \leq i, j \leq n$ with $j \neq s, t$. Hence, we have edge relations

- $(E_{ij})_{e_s} = (E_{ij})_{e_t}$,
- $(F_j)_{e_s} = (F_j)_{e_t}$.

Since we already obtained presentations of $\Gamma_2(2)$ and $\Gamma_2(3)$, from Lemma 6.1 and Remark 4.4, we obtain the presentation of $\Gamma_2(n)$ for $n \geq 4$, by induction on n .

Thus, we complete the proof of Theorem 1.1.

APPENDIX A.

In this section, we check Tietze transformations of Subsection 4.3.

Let $\widehat{\Gamma}$ denote the quotient of $\bigstar_{1 \leq i \leq 7} \Gamma_2(3)_{v_i}$ by the normal subgroup generated by edge relators. By the edge relations of Subsection 4.3, we have the following relations, in $\widehat{\Gamma}$,

- (1)
 - $(E_{23})_{v_2} = (E_{23})_{v_1}$,
 - $(E_{13})_{v_2} = (E_{13})_{v_1}$,
 - $(F_3)_{v_2} = (F_3)_{v_1}$,
- (2)
 - $(E_{31})_{v_3} = (E_{31})_{v_2}$,
 - $(E_{32})_{v_3} = (E_{32})_{v_1}$,

- $(E_{12})_{v_3} = (E_{12})_{v_1}$,
- $(E_{21})_{v_3} = (E_{21})_{v_2}$,
- $(F_1)_{v_3} = (F_1)_{v_2}$,
- $(F_2)_{v_3} = (F_2)_{v_1}$,
- (3) • $(E_{21}F_2E_{12}F_1)_{v_4} = (E_{21})_{v_2}(F_2)_{v_1}(E_{12})_{v_1}(F_1)_{v_2}$,
- $(E_{13}E_{23})_{v_4} = (E_{13})_{v_1}(E_{23})_{v_1}$,
- $(E_{23})_{v_4} = (E_{23})_{v_1}$,
- $(E_{31}^{-1}E_{32})_{v_4} = (E_{31})_{v_2}^{-1}(E_{32})_{v_1}$,
- $(E_{21}F_2)_{v_4} = (E_{21})_{v_2}(F_2)_{v_1}$,
- $(F_3)_{v_4} = (F_3)_{v_1}$,
- (4) • $(E_{31}F_3E_{13}F_1)_{v_5} = (E_{31})_{v_2}(F_3)_{v_1}(E_{13})_{v_1}(F_1)_{v_2}$,
- $(E_{12}E_{32})_{v_5} = (E_{12})_{v_1}(E_{32})_{v_1}$,
- $(E_{32})_{v_5} = (E_{32})_{v_1}$,
- $(E_{21}^{-1}E_{23})_{v_5} = (E_{21})_{v_2}^{-1}(E_{23})_{v_1}$,
- $(E_{31}F_3)_{v_5} = (E_{31})_{v_2}(F_3)_{v_1}$,
- $(F_2)_{v_5} = (F_2)_{v_1}$,
- (5) • $(E_{32}F_3E_{23}F_2)_{v_6} = (E_{32})_{v_1}(F_3)_{v_1}(E_{23})_{v_1}(F_2)_{v_1}$,
- $(E_{21}E_{31})_{v_6} = (E_{21})_{v_2}(E_{31})_{v_2}$,
- $(E_{31})_{v_6} = (E_{31})_{v_2}$,
- $(E_{12}^{-1}E_{13})_{v_6} = (E_{12})_{v_1}^{-1}(E_{13})_{v_1}$,
- $(E_{32}F_3)_{v_6} = (E_{32})_{v_1}(F_3)_{v_1}$,
- $(F_1)_{v_6} = (F_1)_{v_2}$,
- (6) • $(E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})_{v_7} = (E_{21})_{v_2}(F_2)_{v_1}(E_{12})_{v_1}(F_1)_{v_2}(E_{31})_{v_2}^{-1}(E_{32})_{v_1}$,
- $(E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})_{v_7} = (E_{31})_{v_2}(F_3)_{v_1}(E_{13})_{v_1}(F_1)_{v_2}(E_{21})_{v_2}^{-1}(E_{23})_{v_1}$,
- $(E_{21}^{-1}E_{23})_{v_7} = (E_{21})_{v_2}^{-1}(E_{23})_{v_1}$,
- $(E_{31}^{-1}E_{32})_{v_7} = (E_{31})_{v_2}^{-1}(E_{32})_{v_1}$,
- $(E_{21}F_2)_{v_7} = (E_{21})_{v_2}(F_2)_{v_1}$,
- $(E_{31}F_3)_{v_7} = (E_{31})_{v_2}(F_3)_{v_1}$.

Using Tietze transformations, we obtain a presentation of $\widehat{\Gamma}$ whose generators are $(E_{12})_{v_1}$, $(E_{13})_{v_1}$, $(E_{23})_{v_1}$, $(E_{32})_{v_1}$, $(F_2)_{v_1}$, $(F_3)_{v_1}$, $(E_{21})_{v_2}$, $(E_{31})_{v_2}$ and $(F_1)_{v_2}$. To avoid complication of notations, we rewrite $X = X_{v_i}$. Then we have a finite presentation of $\widehat{\Gamma}$ with generators E_{12} , E_{13} , E_{23} , E_{32} , F_2 , F_3 , E_{21} , E_{31} and F_1 , and with the following relators

- (1.1) F_2^2, F_3^2 ,
- (1.2) $(E_{12}F_2)^2, (E_{13}F_3)^2, (E_{23}F_2)^2, (E_{23}F_3)^2, (E_{32}F_2)^2, (E_{32}F_3)^2, (F_2F_3)^2$,
- (1.3) $[E_{12}, E_{13}], [E_{12}, E_{32}], [E_{12}, F_3], [E_{13}, E_{23}], [E_{13}, F_2], [E_{23}, E_{12}]E_{13}^2, [E_{32}, E_{13}]E_{12}^2$,
- (2.1) F_1^2 ,
- (2.2) $(E_{13}F_1)^2, (E_{21}F_1)^2, (E_{31}F_1)^2, (E_{31}F_3)^2, (F_1F_3)^2$,
- (2.3) $[E_{21}, E_{23}], [E_{21}, E_{31}], [E_{21}, F_3], [E_{23}, F_1], [E_{13}, E_{21}]E_{23}^2, [E_{31}, E_{23}]E_{21}^2$,
- (3.2) $(E_{12}F_1)^2, (E_{21}F_2)^2, (F_1F_2)^2$,
- (3.3) $[E_{31}, E_{32}], [E_{31}, F_2], [E_{32}, F_1], [E_{12}, E_{31}]E_{32}^2, [E_{21}, E_{32}]E_{31}^2$,
- (4.3) $[E_{31}^{-1}E_{32}, E_{13}E_{23}](E_{21}F_2E_{12}F_1)^2$,
- (5.3) $[E_{21}^{-1}E_{23}, E_{12}E_{32}](E_{31}F_3E_{13}F_1)^2$,
- (6.3) $[E_{12}^{-1}E_{13}, E_{21}E_{31}](E_{32}F_3E_{23}F_2)^2$,
- (7.3) (a) $[E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32}, E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23}]$,
- (b) $[E_{21}^{-1}E_{23}, E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32}](E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23})^2$,
- (c) $[E_{31}^{-1}E_{32}, E_{31}F_3E_{13}F_1E_{21}^{-1}E_{23}](E_{21}F_2E_{12}F_1E_{31}^{-1}E_{32})^2$.

Let X , Y and Z be

$$\begin{aligned} X &= \{(F_i F_j)^2, (E_{ij} F_i)^2, (E_{ij} F_j)^2, [E_{ij}, F_k] \mid \{i, j, k\} = \{1, 2, 3\}\}, \\ Y &= \{[E_{ij}, E_{ik}], [E_{ij}, E_{kj}] \mid \{i, j, k\} = \{1, 2, 3\}\}, \\ Z &= \{[E_{ij}, E_{ki}] E_{kj}^2 \mid \{i, j, k\} = \{1, 2, 3\}\}. \end{aligned}$$

We show that relators (4.3), (5.3), (6.3) and (b), (c) of (7.3) are obtained from relators X , Y , Z and (a) of (7.3). In transformation, the notation “ \equiv ” means conjugation. An underline means applying relators Y , Z or (a) of (7.3).

Lemma A.1. *Under relators (1.-), (2.-), (3.-) and conjugation,*

- (1) *the relator (a) of (7.3) is equivalent to the relator $(E_{j1} E_{1j}^{-1} E_{kj}^{-1} E_{jk} E_{1k} E_{k1}^{-1})^2$,*
- (2) *relators (b) and (c) of (7.3) are equivalent to the relator*
 $E_{kj}^{-1} E_{1j} E_{j1}^{-1} E_{jk}^{-1} E_{kj} E_{1j}^{-1} E_{j1} E_{jk} E_{1k}^{-1} E_{k1} E_{1k}^{-1} E_{k1},$

where $(j, k) = (2, 3)$ or $(3, 2)$.

Proof. (1) At first, we delete words F_1 , F_2 and F_3 , using relators X , and then transform as follows.

$$\begin{aligned} & [E_{j1} F_j E_{1j} F_1 E_{k1}^{-1} E_{kj}, E_{k1} F_k E_{1k} F_1 E_{j1}^{-1} E_{jk}] \\ &= (E_{j1} F_j E_{1j} F_1 \underline{E_{k1}^{-1} E_{kj}}) (E_{k1} F_k E_{1k} F_1 E_{j1}^{-1} E_{jk}) \\ & \quad \cdot (E_{kj}^{-1} E_{k1} F_1 E_{1j}^{-1} F_j \underline{E_{j1}^{-1}}) (\underline{E_{jk}^{-1} E_{j1} F_1 E_{1k}^{-1} F_k E_{k1}^{-1}}) \\ &= \underset{X}{E_{j1} E_{1j}^{-1} E_{kj}^{-1} \cdot \underline{E_{1k} E_{j1} E_{jk}} \cdot E_{kj}^{-1} E_{k1}^{-1} E_{1j}^{-1} \cdot E_{jk} E_{1k} E_{k1}^{-1}} \\ &= E_{j1} E_{1j}^{-1} E_{kj}^{-1} \cdot E_{jk} E_{1k} E_{j1} \cdot \underline{E_{kj}^{-1} E_{k1}^{-1} E_{1j}^{-1}} \cdot E_{jk} E_{1k} E_{k1}^{-1} \\ &= E_{j1} E_{1j}^{-1} E_{kj}^{-1} \cdot E_{jk} E_{1k} E_{k1}^{-1} E_{j1} \cdot \underline{E_{kj}^{-1} E_{1j}^{-1}} \cdot E_{jk} E_{1k} E_{k1}^{-1} \\ &= E_{j1} E_{1j}^{-1} E_{kj}^{-1} E_{jk} E_{1k} E_{k1}^{-1} \cdot E_{j1} E_{1j}^{-1} E_{kj}^{-1} E_{jk} E_{1k} E_{k1}^{-1} \\ &= (E_{j1} E_{1j}^{-1} E_{kj}^{-1} E_{jk} E_{1k} E_{k1}^{-1})^2. \end{aligned}$$

Thus, we obtain the claim.

(2) Similarly, we delete words F_1 , F_2 and F_3 as follows.

$$\begin{aligned} & [E_{j1}^{-1} E_{jk}, E_{j1} F_j E_{1j} F_1 E_{k1}^{-1} E_{kj}] \\ &= E_{jk}^{-1} E_{j1} \cdot \underline{E_{kj}^{-1} E_{k1} F_1 E_{1j}^{-1} F_j E_{j1}^{-1}} \cdot \underline{E_{j1}^{-1} E_{jk} \cdot E_{j1} F_j E_{1j} F_1 E_{k1}^{-1} E_{kj}} \\ &= \underset{X}{E_{jk}^{-1} E_{j1} \cdot E_{k1} E_{kj}^{-1} E_{1j} E_{j1}^{-1} \cdot E_{jk}^{-1} \cdot E_{1j}^{-1} E_{k1}^{-1} E_{kj}}, \\ & \quad (E_{k1} F_k E_{1k} F_1 E_{j1}^{-1} E_{jk})^2 \\ &= E_{k1} F_k E_{1k} F_1 E_{j1}^{-1} E_{jk} \cdot E_{k1} F_k E_{1k} F_1 E_{j1}^{-1} E_{jk} \\ &= \underset{X}{E_{k1} E_{1k}^{-1} E_{j1} E_{jk}^{-1} \cdot E_{k1} E_{1k}^{-1} E_{j1}^{-1} E_{jk}}. \end{aligned}$$

We next calculate

$$\begin{aligned}
& [E_{j1}^{-1}E_{jk}, E_{j1}F_jE_{1j}F_1E_{k1}^{-1}E_{kj}](E_{k1}F_kE_{1k}F_1E_{j1}^{-1}E_{jk})^2 \\
&= E_{jk}^{-1}E_{j1}E_{k1}E_{kj}^{-1}E_{1j}E_{j1}^{-1}E_{jk}^{-1} \underbrace{E_{1j}^{-1}E_{k1}^{-1}E_{kj}}_Y \cdot E_{k1} \underbrace{E_{1k}^{-1}E_{j1}E_{jk}^{-1}}_Z \\
&\quad \cdot E_{k1}E_{1k}^{-1}E_{j1}^{-1}E_{jk} \\
&\equiv E_{kj}^{-1}E_{1j}E_{j1}^{-1}E_{jk}^{-1}E_{kj}E_{1j}^{-1}E_{j1}E_{jk}E_{1k}^{-1}E_{k1}E_{1k}^{-1}E_{k1}.
\end{aligned}$$

Thus, we obtain the claim. \square

Proposition A.2. *Each of relators (b) and (c) of (7.3) is obtained from relators X , Y , Z and (a) of (7.3).*

Proof. Let $(j, k) = (2, 3)$ or $(3, 2)$. We calculate

$$\begin{aligned}
1 &= E_{j1}E_{1j}^{-1}E_{kj}^{-1} \underbrace{E_{jk}E_{1k}E_{k1}^{-1}}_Y \cdot E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1} \\
&= E_{j1} \underbrace{E_{1j}^{-1}E_{kj}^{-1}E_{1k}}_Z \underbrace{E_{jk}E_{k1}^{-1}E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}}_Z \\
&= E_{j1}E_{1k} \underbrace{E_{1j}E_{kj}^{-1}E_{k1}^{-1}E_{j1}^{-1}E_{jk}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}}_Z \\
&= E_{j1}E_{1k}E_{k1}^{-1}E_{kj}E_{1j}E_{j1}^{-1}E_{jk}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1} \\
&\equiv E_{kj}E_{1j}E_{j1}^{-1}E_{jk} \underbrace{E_{1j}^{-1}E_{kj}^{-1}}_Y E_{jk}E_{1k} \underbrace{E_{k1}^{-1}E_{j1}E_{1k}E_{k1}^{-1}}_Y \\
&= E_{kj}E_{1j}E_{j1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1} \underbrace{E_{jk}E_{1k}E_{j1}E_{k1}^{-1}}_Z E_{1k}E_{k1}^{-1} \\
&= (E_{jk}E_{1k}E_{k1}^{-1}E_{k1}E_{1k}E_{jk}^{-1}) \underbrace{E_{kj}E_{1j}E_{j1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1}}_{(a) \text{ of (7.3)}} \\
&= E_{jk}E_{1k}E_{k1}^{-1}E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1} \\
&\equiv E_{kj}^{-1} \cdot E_{jk}E_{1k}E_{k1}^{-1}E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1} (E_{j1}E_{1j}^{-1}E_{1j}E_{j1}^{-1}) \\
&\quad \cdot E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1} \cdot E_{kj} \\
&= \underbrace{(E_{kj}^{-1}E_{jk}E_{1k}E_{k1}^{-1}E_{j1}E_{1j}^{-1})^2}_{(a) \text{ of (7.3)}} E_{1j}E_{j1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1}E_{kj} \\
&= E_{1j}E_{j1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1}E_{kj} \\
&\equiv F_k \cdot E_{kj}E_{1j}E_{j1}^{-1}E_{jk}E_{kj}^{-1}E_{1j}^{-1}E_{j1}E_{jk}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{k1}^{-1} \cdot F_k \\
&\underset{X}{=} E_{kj}^{-1}E_{1j}E_{j1}^{-1}E_{jk}^{-1}E_{kj}E_{1j}^{-1}E_{j1}E_{jk}E_{1k}^{-1}E_{k1}E_{1k}^{-1}E_{k1}.
\end{aligned}$$

By Lemma A.1, we obtain the claim. \square

Proposition A.3. *Each of relators (4.3), (5.3) and (6.3) is obtained from other relators and conjugation.*

Proof. We first consider relators (4.3) and (5.3). Let $(j, k) = (2, 3)$ or $(3, 2)$.

$$\begin{aligned}
& [E_{j1}^{-1}E_{jk}, E_{1j}E_{kj}](E_{k1}F_kE_{1k}F_1)^2 \\
&= E_{jk}^{-1}E_{j1} \cdot \underbrace{E_{kj}^{-1}E_{1j}^{-1}}_Y \cdot \underbrace{E_{j1}^{-1}E_{jk}}_Y \cdot E_{1j}E_{kj} \cdot E_{k1}F_kE_{1k}F_1 \cdot E_{k1}F_kE_{1k}F_1 \\
&\stackrel{X}{=} E_{jk}^{-1}E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{j1}^{-1}E_{1j}E_{kj}E_{k1}E_{1k}^{-1}E_{k1}E_{1k}^{-1} \\
&\equiv F_1(E_{k1}E_{1k}^{-1}E_{k1}E_{1k}^{-1}E_{jk}^{-1}E_{j1}E_{1j}^{-1}E_{kj}^{-1}E_{jk}E_{j1}^{-1}E_{1j}E_{kj})F_1 \\
&\stackrel{X}{=} E_{k1}^{-1}E_{1k}E_{k1}^{-1}E_{1k}E_{jk}^{-1}E_{j1}^{-1}E_{1j}E_{kj}^{-1}E_{jk}E_{j1}^{-1}E_{1j}E_{kj} \\
&= (E_{kj}^{-1}E_{1j}E_{j1}^{-1}E_{jk}^{-1}E_{kj}E_{1j}^{-1}E_{j1}E_{jk}E_{1k}^{-1}E_{k1}E_{1k}^{-1}E_{k1})^{-1}.
\end{aligned}$$

We next consider the relator (6.3).

$$\begin{aligned}
& [E_{12}^{-1}E_{13}, E_{21}E_{31}](E_{32}F_3E_{23}F_2)^2 \\
&= E_{13}^{-1}E_{12} \cdot E_{31}^{-1}E_{21}^{-1} \cdot E_{12}^{-1}E_{13} \cdot \underbrace{E_{21}E_{31} \cdot E_{32}F_3E_{23}F_2 \cdot E_{32}F_3E_{23}F_2}_Z \\
&\stackrel{X}{=} E_{13}^{-1}E_{12}E_{31}^{-1}E_{21}^{-1}E_{12}^{-1}E_{13}E_{31}^{-1}E_{32}\underbrace{E_{21}E_{23}^{-1}}_YE_{32}E_{23}^{-1} \\
&\equiv \underbrace{E_{23}^{-1}E_{13}^{-1}E_{12}E_{31}^{-1}E_{21}^{-1}E_{12}^{-1}E_{13}E_{31}^{-1}E_{32}E_{23}^{-1}E_{21}E_{32}}_Z \\
&= E_{13}E_{12}\underbrace{E_{23}^{-1}E_{31}^{-1}E_{21}^{-1}E_{12}^{-1}E_{13}E_{31}^{-1}E_{32}E_{23}^{-1}E_{21}E_{32}}_Z \\
&= E_{13}E_{12}E_{31}^{-1}E_{21}\underbrace{E_{23}^{-1}E_{12}^{-1}E_{13}E_{31}^{-1}E_{32}E_{23}^{-1}E_{21}E_{32}}_Z \\
&= E_{13}E_{12}E_{31}^{-1}E_{21}E_{12}^{-1}E_{23}^{-1}E_{13}^{-1}E_{31}^{-1}E_{32}E_{23}^{-1}E_{21}E_{32} \\
&\equiv \underbrace{E_{12}^{-1}E_{23}^{-1}E_{13}^{-1}E_{31}^{-1}E_{32}E_{23}^{-1}E_{21}\underbrace{E_{32}E_{13}E_{12}E_{31}^{-1}}_ZE_{21}}_Z \\
&= E_{23}^{-1}E_{13}\underbrace{E_{12}^{-1}E_{31}^{-1}E_{32}}_Z\underbrace{E_{23}^{-1}E_{21}E_{13}}_Z\underbrace{E_{32}E_{12}^{-1}E_{31}^{-1}E_{21}}_Z \\
&= E_{23}^{-1}E_{13}E_{31}^{-1}E_{32}^{-1}\underbrace{E_{12}^{-1}E_{23}E_{13}}_Z\underbrace{E_{21}E_{32}^{-1}E_{31}^{-1}E_{12}^{-1}E_{21}}_Z \\
&= E_{23}^{-1}E_{13}E_{31}^{-1}E_{32}^{-1}E_{23}E_{13}^{-1}\underbrace{E_{12}^{-1}E_{31}E_{32}^{-1}}_ZE_{21}E_{12}^{-1}E_{21} \\
&= E_{23}^{-1}E_{13}E_{31}^{-1}E_{32}^{-1}E_{23}E_{13}^{-1}E_{31}E_{32}E_{12}^{-1}E_{21}E_{12}^{-1}E_{21}.
\end{aligned}$$

By Lemma A.1, each of relators (4.3), (5.3) and (6.3) is obtained from relators (1.-), (2.-), (3.-) and (b), (c) of (7.3). Thus, we obtain the claim. \square

ACKNOWLEDGEMENT

The author would like to express his thanks to Dan Margalit, Andrew Putman and Neil Fullarton for informing the author about their results including their finite presentations of $\Gamma_2(n)$, Susumu Hirose and Masatoshi Sato for their valuable suggestions and useful comments.

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